I - INTRODUCTION: A GEOMETRIC VIEW

I.1 Coxeter systems

A Coxeter system ("of type m") is a pair (W, S) where S is a (for us) finite set, and W is a group presented as

$$W := \langle S \mid (ss')^{m(s,s')} = id \rangle, \qquad (*)$$

where

$$m : S \times S \longrightarrow \{1, 2, 3, \ldots\} \cup \{\infty\}$$

is a function such that

$$m(s,s') = m(s',s) \quad \text{for all } s, s' \in S, \text{ and}$$

$$m(s,s') = id \iff s = s'.$$

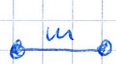
I.1.1 Terminology.

The <u>Coxeter matrix</u> of (W,S) is $(m(s,s'))_{s,s' \in S}$

The <u>Coxeter graph</u> of (W,S) has S as set of vertices, and edge-set $\{\{s,s'\} \mid m(s,s') \geq 3\}$. Edges

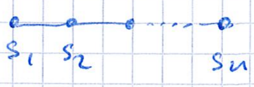
We label an edge $\{s,s'\}$ with $m(s,s')$ if $m(s,s') \geq 4$.

<u>Examples</u>:

$$\begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix} \longleftrightarrow$$ 

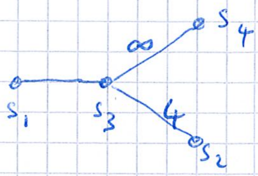
$; \quad W \cong I_2(m)$ "dihedral"

$$\begin{pmatrix} 1 & 3 & 2 & \cdots & \cdots & 2 \\ 3 & 1 & 3 & & & \vdots \\ 2 & 3 & 1 & 3 & & 2 \\ \vdots & & 3 & 1 & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 3 \\ 2 & \cdots & \cdots & 2 & 3 & 1 \end{pmatrix} \longleftrightarrow$$



; $W = A_n \simeq S_{n+1}$ "symmetric group"

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 1 & 4 & 2 \\ 3 & 4 & 1 & \infty \\ 2 & 2 & \infty & 1 \end{pmatrix} \longleftrightarrow$$



The precise meaning of $(*)$ can be espressed by the following "universality property":

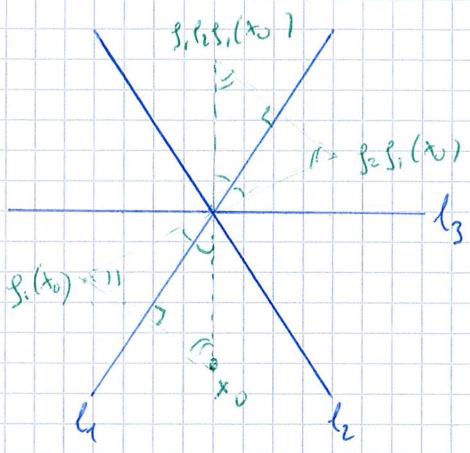
"If G is a group and $f: S \to G$ is a function with $\left(f(s)f(s')\right)^{m(s,s')} = id$ for all $s, s' \in S$, $m(s,s') < \infty$, then there is a unique extension of f to a group homomorphism $\hat{f}: W \to G$

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, hook] & G \\ W \arrow[ur, "\exists!\hat{f}"'] & \end{tikzcd}$$

I.1.2. Intuition

The definition of Coxeter groups and Coxeter systems was motivated by the study of finite groups generated by (orthogonal) reflections.

For example, consider three equiangular lines l_1, l_2, l_3 in $\mathbb{R}^2$



Let ρ_i be the orthogonal reflection w.r.t. the line l_i.

If n_i is a unit normal vector for l_i:

$$\rho_i(x) = x - 2\langle x, n_i\rangle n_i \qquad \forall x \in \mathbb{R}^2$$

Clearly $\rho_i^2 = id$ for $i = 1, 2, 3$.

Moreover:

- $\rho_1 \circ \rho_2 \circ \rho_1(x) = x$ for all $x \in l_3$
- for x_0 on the bisector of l_1, l_2:

  $\rho_1 \circ \rho_2 \circ \rho_1(x_0) = -x_0 = \rho_3(x_0)$

$\Big\}$ See picture, or compute.

therefore: $\rho_1 \circ \rho_2 \circ \rho_1 = \rho_3$ (analogously $\rho_2 \circ \rho_1 \circ \rho_2 = \rho_3$)

hence: $(\rho_1 \circ \rho_2)^3 = id$

The universality property gives us a group homomorphism

$$\begin{aligned} \rho : A_2 \simeq I_2(3) &\longrightarrow GL(\mathbb{R}^2) \\ s_1 &\longmapsto \rho_1 \\ s_2 &\longmapsto \rho_2 \end{aligned}$$

This map is in fact injective and gives us a "representation by (orthogonal) reflections" of A_2

In general, there is no hope to represent all (W, S) as groups generated by <u>orthogonal</u> reflections. But we can do so by dropping the orthogonality condition.

I.2 Linear representations of Coxeter systems

Let (W,S) be a Coxeter system.

Define
$$V := \bigoplus_{s\in S} \mathbb{R}$$

and let $\alpha_s := (0,\dots,0,\underset{\substack{\uparrow\\ s\text{-component}}}{1},0,\dots,0) \in V$ for all $s\in S$.

We "impose" a geometry on V through the symmetric bilinear form B defined on basis vectors as

$$B(\alpha_s,\alpha_{s'}) := -\cos\left(\frac{\pi}{m(s,s')}\right) \quad \text{for all } s,s'\in S$$

where we think "$\frac{\pi}{\infty} = 0$", thus set $B(\alpha_s, \alpha_{s'}) = -1$ if $m(s,s') = \infty$.

For each $s\in S$ define now a linear map $\rho_s : V \to V$ by setting
$$\rho_s(x) := x - 2B(\alpha_s, x)\alpha_s \qquad (**)$$
for all $x\in V$.

I.2.1 Remark.

For every $s\in S$ we have $B(\alpha_s,\alpha_s) = 1$, so α_s is anisotropic and the subspace of "orthogonals to α_s w.r.t. B",
$$F_s := \{ x\in V \mid B(x,\alpha_s) = 0 \}$$
is complementary to $\mathbb{R}\alpha_s$: $V = \mathbb{R}\alpha_s \oplus F_s$.

Now we easily see:
- $\rho_s(x) = x$ for all $x\in F_s$
- since moreover $\rho_s(\alpha_s) = -\alpha_s$, we have $\rho_s^2 = \mathrm{id}$.

Thus ρ_s has order 2 and fixes a hyperplane: this is our analog / generalization of a "orthogonal reflection".

In fact ρ_s preserves the form B:

$$\begin{aligned} B(\rho_s(x), \rho_s(x)) &= B(x - 2B(x,\alpha_s)\alpha_s,\ x - 2B(x,\alpha_s)\alpha_s) \\ &= B(x,x) - 4B(x,\alpha_s)B(x,\alpha_s) + 4B(x,\alpha_s)^2 B(\alpha_s,\alpha_s) \\ &= B(x,x) \quad \text{for all } x \in V. \end{aligned}$$

We just can't say the word "orthogonal" because B needs not be positive definite.

I.2.2. Theorem.

There exists a unique group homomorphism

$$\rho : W \longrightarrow GL(V)$$

with $\rho(s) = \rho_s$ for all $s \in S$. The group $\rho(W)$ preserves the form B and for all $s, s' \in S$ the order of ss' in W is exactly $m(s,s')$

Proof: Notice that it is enough to prove that the order of $\rho_s \circ \rho_{s'}$ is exactly $m(s,s')$ whenever $s \neq s'$.

In fact this implies
- existence & uniqueness by the Universality property
- the claim on the order of ss', trivially.

That $\rho(W)$ preserves B follows from Remark I.2.1.

In order to prove our claim, fix $s \neq s' \in S$ and consider

$$U := \mathbb{R}\alpha_s \oplus \mathbb{R}\alpha_{s'} \subseteq V$$

- The restriction $B_{|U}$ of B on U is positive semidefinite

Pf: For any $u = a\alpha_s + b\alpha_{s'}$, $a, b \in \mathbb{R}$, compute

$$\begin{aligned} B(u,u) &= a^2 B(\alpha_s,\alpha_s) + 2ab\, B(\alpha_s,\alpha_{s'}) + b^2 B(\alpha_{s'},\alpha_{s'}) \\ &= \left(a - b\cos\left(\frac{\pi}{m(s,s')}\right)\right)^2 + b^2 \sin^2\left(\frac{\pi}{m(s,s')}\right) \geq 0 \end{aligned}$$

$\lrcorner$

- $B|_U$ is degenerate exactly if $m(s,s') < \infty$

  <u>Pf</u>: If $m(s,s') < \infty$, then $\sin^2\left(\frac{\pi}{m(s,s')}\right) > 0$, so $B(u,u) > 0$ $\forall u$.

  If $m(s,s') = \infty$, for $u := \alpha_s + \alpha_{s'}$ we compute $B(u,u) = 0$ $\lrcorner$

- $\rho_s(U) \subseteq U$ and $\rho_{s'}(U) \subseteq U$. In particular, $\rho_s \circ \rho_{s'}$ defines an operator $U \to U$.

  Pf: By inspection of the definition $(**)$. $\lrcorner$

<u>Case 1: $m(s,s') < \infty$</u> Here $B|_U$ is positive definite and we can compute the matrix of $\rho_s \circ \rho_{s'}$ w.r.t. an B_U-orthonormal basis. Let $m := m(s,s')$.

E.g., let $\beta := \frac{\cos \pi/m}{\sin \pi/m} \alpha_s + \frac{1}{\sin \pi/m} \alpha_{s'}$.

Then, $B(\alpha_s, \beta) = 0$, $B(\beta,\beta) = 1$. Thus the basis $\{\alpha_s, \beta\}$ is orthonormal w.r.t. $B|_U$. We compute

$$\rho_s(\alpha_s) = -\alpha_s \qquad ; \qquad \rho_s(\beta) = \beta \quad ;$$

$$\rho_{s'}(\alpha_s) = \alpha_s - 2B(\alpha_{s'}, \alpha_s)\alpha_{s'} = \alpha_s + 2\cos\tfrac{\pi}{m}\alpha_{s'} = \dots$$

$$\dots = \left(1 - 2\cos^2\tfrac{\pi}{m}\right)\alpha_s + 2\sin\tfrac{\pi}{m}\cos\tfrac{\pi}{m}\beta$$

$$= -\cos 2\tfrac{\pi}{m}\,\alpha_s + \sin 2\tfrac{\pi}{m}\,\beta$$

$$\rho_{s'}(\beta) = \beta - 2B(\alpha_{s'}, \beta)\alpha_{s'} = \beta - 2B\left(\sin\tfrac{\pi}{m}\beta - \cos\tfrac{\pi}{m}\alpha_s, \beta\right)\alpha_{s'} =$$

$$\dots = \left(1 - 2\sin^2\tfrac{\pi}{m}\right)\beta + 2\sin\tfrac{\pi}{m}\cos\tfrac{\pi}{m}\alpha_s =$$

$$= \cos 2\tfrac{\pi}{m}\,\beta + \sin 2\tfrac{\pi}{m}\,\alpha_s$$

$\lrcorner$

With respect to this basis:

$$\underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{\rho_s} \underbrace{\begin{pmatrix} -\cos 2\frac{\pi}{m} & \sin 2\frac{\pi}{m} \\ \sin 2\frac{\pi}{m} & \cos 2\frac{\pi}{m} \end{pmatrix}}_{\rho_{s'}} = \underbrace{\begin{pmatrix} \cos 2\frac{\pi}{m} & -\sin 2\frac{\pi}{m} \\ \sin \frac{2\pi}{m} & \cos \frac{2\pi}{m} \end{pmatrix}}_{\rho_s \circ \rho_{s'}}$$

Thus, up to a change of scalar product, $\rho_s \circ \rho_{s'}$ has the form of a rotation by $\frac{2\pi}{m}$, thus has order exactly m.

Since
$$V = U \oplus (F_s \cap F_{s'})$$
and $\rho_s \circ \rho_{s'}$ fixes $F_s \cap F_{s'}$ pointwise, we have
$$\rho_s \circ \rho_{s'}(u \oplus f) = \rho_s \circ \rho_{s'}(u) \oplus f$$
and $\rho_s \circ \rho_{s'}$ has order exactly m also on V ⌋

Case 2: $m(s,s') = \infty$

Here $B(\alpha_s, \alpha_{s'}) = -1$ and, for $\gamma = \alpha_s + \alpha_{s'}$ we compute
$$B(\gamma, \alpha_s) = B(\gamma, \alpha_{s'}) = 0, \quad \text{hence} \quad \rho_s(\gamma) = \rho_{s'}(\gamma) = \gamma.$$
With this:
$$\rho_s \circ \rho_{s'}(\alpha_s) = \rho_s(\alpha_s - 2B(\alpha_{s'}, \alpha_s)\alpha_{s'}) = \ldots$$
$$\ldots = 2\gamma + \alpha_s,$$
hence $(\rho_s \circ \rho_{s'})^k(\alpha_s) = 2k\gamma + \alpha_s$, thus $\rho_s \circ \rho_{s'}$ has infinite order on U (and hence also on V).

☒

I.2.3 Corollary If $s, s' \in S$, $s \neq s'$ in S implies $s \neq s'$ in W.

Proof ~~$ss' = id$~~ $s = s'$ in W implies $\rho_s \circ \rho_{s'} = \rho_{id} = id$ in $GL(V)$, thus the order of $\rho_s \circ \rho_{s'}$ is 1. By the theorem, this order must equal $m(s,s')$; but $m(s,s') = 1$ implies $s = s'$ by definition

☒

I.3 Lengths and roots

In order to prove that the representation ρ constructed in I.2. is "faithful" we need to check that $\rho: W \to GL(V)$ is injective, i.e., its kernel is trivial. We already know that $\rho_s \neq id$ $\forall s \in S$: in other words, no "word of length 1" is in the kernel of ρ.

Idea: prove $\rho_w \neq id$ for all $w \in W \setminus \{id\}$ by induction on "how complicated a word in S w can be".

As a measure of this complexity, we introduce the length of any $w \in W$ as defined by

$$\ell(w) := \min\{k \in \mathbb{N} \mid w = s_1 \cdots s_k \text{ for } s_i \in S\}$$

so that, e.g., $\ell(s) = 1$ $\forall s \in S$; $\ell(w) = 0 \Leftrightarrow w = id$.

It is in general hard to know the length of a product of elements from the lengths of the factors.

Example: Let $W = A_2$, write $S = \{s_1, s_2\}$. For $w := s_1$:

- $\ell(ws_1) = \ell(s_1^2) = 0 < \ell(w)$
- $\ell(ws_2) = \ell(s_1 s_2) = 2 > \ell(w)$

In order to understand the situation we introduce the root system associated to (W, S), defined as

$$\Phi := \{\rho_w(\alpha_s) \mid w \in W, s \in S\} \subseteq V$$

Elements of Φ are called roots.

I.3.1 Remarks and definition

- Roots are unit vectors (w.r.t. B)
- Because for all $s \in S$ $\rho_s(\alpha_s) = -\alpha_s$, we have $\Phi = -\Phi$.
- Since $\{\alpha_s \mid s \in S\}$ is a basis for V, there is a unique expression

$$\alpha = \sum_{s \in S} c_s^\alpha \alpha_s \qquad \text{with } c_s^\alpha \in \mathbb{R}$$

for every $\alpha \in \Phi$.

We call a root $\alpha \in \Phi$ positive (resp. negative) if $c_s^\alpha \geq 0$ for all $s \in S$ (resp. $c_s^\alpha \leq 0$). In this case we will write "$\alpha > 0$" (resp. "$\alpha < 0$") for short. Write Φ^+ (Φ^-) for the set of positive (negative) roots.

I.3.2 Theorem.

Let $w \in W$, $s \in S$.

(i) $l(ws) > l(w)$ implies $\rho_w(\alpha_s) > 0$

(ii) $l(ws) < l(w)$ implies $\rho_w(\alpha_s) < 0$

Proof: First of all, notice that (ii) follows from (i):

In fact, $l(ws) < l(w)$ implies $l((ws)s) = l(w) > l(ws)$ and, with (i),
$$0 < \rho_{ws}(\alpha_s) = \rho_w \rho_s(\alpha_s) = -\rho_w(\alpha_s),$$
thus $\rho_w(\alpha_s) < 0$

The proof of (i) is by induction on the length $l(w)$.

- If $l(w) = 0$ then $w = id$ and there is nothing to prove.
- Suppose $l(w) > 0$ and assume the claim holds for all $w' \in W$ with $l(w') < l(w)$

Let s be as in the claim (i.e., $\ell(ws) > \ell(w)$) and choose $s' \in S$ such that $\ell(ws') < \ell(w)$

⌈such s' exists, e.g., choose s' to be the last letter in some minimal-length decomposition of w ⌋

In particular, $s \neq s'$ and we can consider the subgroup W_I of W generated by $I := \{s, s'\}$. Let

$$A := \{ v \in W \mid v^{-1}w \in W_I ,\ \ell(v) + \ell_I(v^{-1}w) = \ell(w) \}$$

where $\ell_I(u)$ is defined for $u \in W_I$ as the minimal length of a word with letters of I representing u.

Clearly,
$$\ell(u) \leq \ell_I(u) \quad \text{for all } u \in W_I \qquad (\Delta)$$

Choose $v \in A$ s.t. $\ell(v)$ is minimal inside A.

- $\underline{\ell(v) < \ell(w)}$

⌈First, $ws' \in A$ because
- $(s'w^{-1})w \in W_I$,
- $\ell(ws') + \ell_I(s') = \ell(w) - 1 + 1 = \ell(w)$.

Thus, by minimality of v, $\ell(v) \leq \ell(ws') = \ell(w) - 1$ ⌋

- $\underline{\ell(vs) > \ell(v)}$

⌈Otherwise,
$$\begin{aligned}
\ell(w) &\leq \ell(vs) + \ell((vs)^{-1}w) \\
&\leq \ell(vs) + \ell_I(sv^{-1}w) \quad \leftarrow (\Delta) \text{ above} \\
&\overset{\text{assumption}}{\leq} \ell(v) - 1 + \ell_I(sv^{-1}w) \\
&\leq \ell(v) - 1 + \ell_I(v^{-1}w) + 1 \\
&= \ell(v) + \ell_I(v^{-1}w) \underset{v \in A}{=} \ell(w)
\end{aligned}$$

and all $\leq$ are equalities, in particular $vs \in A$ contradicting $\ell(vs) < \ell(v)$ ↯ ⌋

Thus we can apply the induction hypothesis to v and s, obtaining $\underline{\underline{f_v(\alpha_s) > 0}}$

By the same argument,

$$\ell(vs') > \ell(v) \quad \text{and} \quad \rho_v(\alpha_{s'}) > 0 \qquad \boxed{\text{Exercise}}$$

We conclude that $\rho_v(a\alpha_s + b\alpha_{s'}) > 0$ for all $a, b \in \mathbb{R}^+$

If we set $v_I := v^{-1}w$, then $w = v v_I$ and the proof is concluded by showing

- $\underline{\rho_{v_I}(\alpha_s) = a\alpha_s + b\alpha_{s'}}$ with $a, b \in \mathbb{R}^+$

⌜ First notice: all reduced expressions vor v_I in W_I end with s'.

⌜ Otherwise $\ell_I(v_I s) < \ell_I(v_I)$, and so

$$\ell(ws) = \ell(vv^{-1}ws) \leq \ell(v) + \ell(v^{-1}ws) = \ell(v) + \ell(v_I s)$$

$$\leq \ell(v) + \ell_I(v_I s) < \ell(v) + \ell_I(v_I) = \ell(w),$$

contrary to the assumption on w ⌟

Now $\underline{\text{suppose } m(s,s') = \infty}$; then $B(\alpha_s, \alpha_{s'}) = -1$, and so

$$\rho_{s'}(\alpha_s) = \alpha_s + 2\alpha_{s'} \quad ; \quad \rho_{ss'}(\alpha_s) = \alpha_s + 2\alpha_{s'} - 2B(\alpha_s + 2\alpha_{s'}, \alpha_s)\alpha_s$$

$$= 3\alpha_s + 2\alpha_{s'}$$

etc.

thus: $\rho_{\underbrace{s \cdots s'}_{2k}}(\alpha_s) = (2k+1)\alpha_s + 2k\alpha_{s'}$

$\rho_{\underbrace{\cdots s'}_{2k-1}}(\alpha_s) = (2k-1)\alpha_s + 2k\alpha_{s'}$ for all $k > 0$,

and the claim holds.

$\underline{\text{If instead } m(s,s') < \infty}$, write $m := m(s,s')$.

In this case $\ell(v_I) < m$ because if $\ell_I(v_I) = m$ then there is a reduced expression for v_I ending in s

⌜

$$v_I = \underbrace{(s')\, s \cdots | ss' | ss'}_{m} = (s) \cdots | s's | s's$$

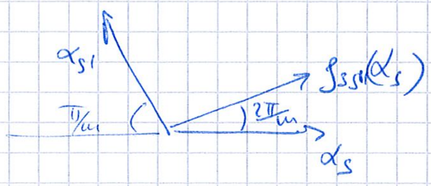
$$\leadsto ss' | \cdots | (s)(s') | \cdots | s' = (ss')^m = id$$ ⌟

contradicting our "first notice" above.

we thus can write

$$v_I = (s')\underbrace{(ss')\cdots(ss')}_{<\frac{m}{2}\text{ pairs}}$$

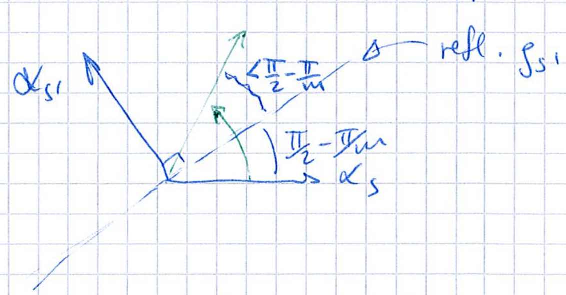
As earlier in the lecture, we know that W_I is dihedral, and thus $\rho_{ss'}$ is "rotation of $\frac{2\pi}{m}$" in a suitable scalar product. I.e. in $\mathbb{R}\alpha_s \oplus \mathbb{R}\alpha_{s'}$ we have



and $<\frac{m}{2}$ times a rotation of $\frac{2\pi}{m}$ will keep α_s inside the positive cone (of angle $\pi - \frac{2\pi}{m}$) spanned by $\alpha_s, \alpha_{s'}$.

If v_I begins with s, we then are done.

If v_I begins with s', after the rotations $\rho_{ss'}$ we have to "reflect at the hyperplane $\alpha_{s'}^{\perp}$"



but the "rotated α_s" will make at most angle $<\frac{\pi}{2}-\frac{\pi}{m}$ with $\alpha_{s'}^{\perp}$, thus be reflected again inside the positive $\alpha_s, \alpha_{s'}$ -cone (α_s has angle $=\frac{\pi}{2}-\pi/m$ with $\alpha_{s'}^{\perp}$)

Thus, in both cases we are done

∎

<u>Corollary</u> The representation $\rho : W \to GL(V)$ is faithful.

<u>Proof</u> Let $w \in \ker \rho$, $w \neq id$. Then there is $s \in S$ with $\ell(ws) < \ell(w)$.

But in this case the Theorem I.3.2. says

$$\rho_w(\alpha_s) < 0.$$

<u>However</u>, since $w \in \ker \rho$, $\rho_w(\alpha_s) = \alpha_s > 0$ $\lightning$

Thus $\ker \rho = \{id\}$ and ρ is injective

$\blacksquare$

I-4 Fundamental domains and the Tits cone

In order to make the geometry more explicit — even in the case when B is not positive definite — we study the "dual" (or "contragredient") representation on V^*, the dual to V, which we endow with the dual basis to $\{\alpha_s \mid s \in S\}$, denoted $\{b_s \mid s \in S\}$.

For $p \in V^*$, $\gamma \in V$ let

$$\langle p, \gamma \rangle \quad \text{"} := p(\gamma)\text{"}$$

denote the standard pairing. Define a representation

$$\rho^*: W \longrightarrow GL(V^*)$$
$$w \longmapsto (\rho_{w^{-1}})^T$$

transpose of the matrix, defines linear function w.r.t. dual basis

We have immediately for all $p \in V^*$, $\gamma \in V$, $w \in W$:

$$\langle \rho^*_w(p), \rho_w(\gamma) \rangle = p^T \rho_{w^{-1}} \rho_w \gamma = p^T \gamma = \langle p, \gamma \rangle$$

written in coordinates

Consider, for $s \in S$, the hyperplane

$$H_s := \{ p \in V^* \mid \langle p, \alpha_s \rangle = 0 \} \subseteq V^*$$

and the half-spaces

$$H_s^+ := \{ p \in V^* \mid \langle p, \alpha_s \rangle > 0 \}$$
$$H_s^- := \{ p \in V^* \mid \langle p, \alpha_s \rangle < 0 \}$$

Define then

$$C := \bigcap_{s \in S} H_s^+$$

I.4.1 Remarks

(i) $C \neq \emptyset$ because for instance $\sum_{s\in S} b_s \in C$

(ii) $\rho^*(x) = x$ for all $x \in H_s$

⌈Proof: Choose a basis of V like: $\alpha_s = \beta_1 ; \underbrace{\beta_2, \dots, \beta_n}_{\text{basis of } Fix(\rho_s)}$

Let $c_1 \dots c_n$ be corresponding dual basis of V^*

Then,

$$\langle \rho_s^*(c_i), \beta_j \rangle = \langle c_i, \rho_s(\beta_j) \rangle = \begin{cases} -1 & \text{if } j = i \\ \langle c_i, \beta_j \rangle & (j \neq i) \end{cases}$$

$$= \begin{cases} -\langle c_i, \alpha_s \rangle & \text{if } j = 1 \\ \langle c_i, \beta_j \rangle & j > 1 \end{cases}$$

Thus $\rho_s^*(c_i)$ evaluates like c_i on $\beta_1 \dots \beta_n$

$\Rightarrow \rho_s^*(c_i) = c_i$ for $i > 1$ ⌟

(iii) H_s^+, H_s^- are open in the std topology of $V^* \simeq \mathbb{R}^n$.

Let

$$D := \overline{C} \qquad \text{(topological closure)}$$

Clearly $\overline{H_s^+} = H_s^+ \cup H_s \; ; \; D = \bigcap_{s\in S} \overline{H_s^+}$

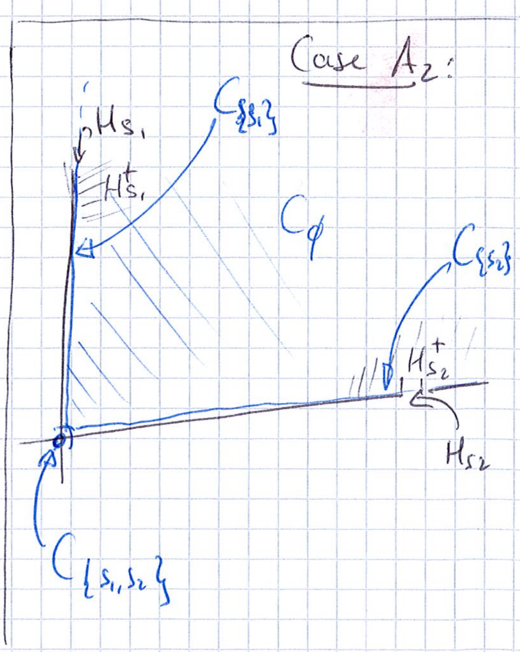
I.4.2. Definition

For each $I \subseteq S$, let

(i) $$C_I := \left(\bigcap_{s\in I} H_s\right) \cap \left(\bigcap_{s\notin I} H_s^+\right)$$

Thus in particular

$$C_\emptyset = C, \quad C_S = \{0\}$$



Moreover, let

(ii) $$U := \bigcup_{w \in W} \rho_w^*(D) \subseteq V^*$$

I.4.3. Remarks

- For all $I \subseteq S$, Remark I.4.1.(ii) implies $\mathrm{stab}(C_I) \supseteq W_I$
- By the definitions,

$$U = \bigcup_{\substack{w \in W \\ I \subseteq S}} \rho_w^*(C_I) \qquad (\star)$$

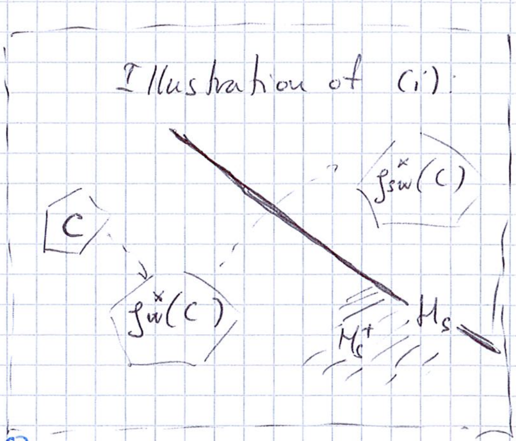
- Also: $\rho_w^*(U) \subseteq U \quad \forall w \in W$.

I.4.4. Lemma

Let $s \in S$, $w \in W$

(i) $\ell(sw) > \ell(w) \iff \rho_w^*(C) \subset H_s^+$

(ii) $\ell(sw) < \ell(w) \iff \rho_w^*(C) \subset H_s^-$



Proof: In order to prove (i) notice first

$$\ell(sw) > \ell(w) \underset{\text{take inverses}}{\iff} \ell(w^{-1}s) > \ell(w^{-1}) \underset{\text{Theorem I.3.2.}}{\iff} \rho_{w^{-1}}(\alpha_s) > 0$$

On the other hand, $\rho_w^*(C) \subseteq H_s^+$ means $\rho_w^*(p) \in H_s^+ \;\; \forall p \in C$.

Equivalently, $0 < \langle \rho_w^*(p), \alpha_s \rangle = \langle p, \rho_{w^{-1}}(\alpha_s) \rangle \quad \forall p \in C$.

This, however, means $\rho_{w^{-1}}(\alpha_s) > 0$

[Otherwise, $\rho_{w^{-1}}(\alpha_s) = \sum_{s \in S} c_s \alpha_s$ with, say, $c_{s'} < 0$ for some $s' \in S$.

but then, for $\varepsilon > 0$ small enough:

$$\langle \underbrace{\Big(\sum_{\substack{s \in S \\ s \neq s'}} \varepsilon b_s\Big) + b_{s'}}_{\in C}, \rho_{w^{-1}}(\alpha_s) \rangle = \varepsilon \Big(\sum_{\substack{s \in S \\ s \neq s'}} c_s\Big) + c_{s'} < 0$$

$\lightning$]

and we have the desired equivalence.
The proof for (ii) is analogous $\square$

I.4.5. Lemma Let $w \in W$ and $s \in S$.

If $\rho^*_w(C_I) \cap C_J \neq \emptyset$, then $I = J$ and $w \in W_I$,

thus $\rho^*_w(C_I) = C_I$.

Corollary. In particular $W_I = \mathrm{stab}(p)$ $\forall p \in C_I$,

and the union in (☆) is disjoint.

Proof of Lemma: By induction on $\ell(w)$.

- If $\ell(w) = 0$, then $w = id$ and there is nothing to prove.
- If $\ell(w) > 0$, choose $s \in S$ with $\ell(sw) < \ell(w)$.

By Lemma I.4.4., $\rho^*_w(C) \subseteq \rho^*_s(H^+_s) = H^-_s$.

Since ρ^*_w is continuous, $\rho^*_w(D) \subseteq \overline{H^-_s}$. Also, by

Remark I.4.3., ρ^*_s fixes every point in the (nonempty) set

$$\rho^*_w(C_I) \cap C_J \subseteq \rho^*_w(D) \cap D \subseteq \overline{H^+_s} \cap \overline{H^-_s} = H_s$$

We conclude:

- ρ^*_s fixes a point p in C_J, thus $s \in J$

  $[\langle p, \alpha_s \rangle = \langle \rho^*_s(p), \rho_s(\alpha_s) \rangle = -\langle p, \alpha_s \rangle \Rightarrow p \in H_s]$

- $C_J \cap \rho^*_{sw}(C_I) = \rho^*_s(C_J \cap \rho^*_w(C_I)) \neq \emptyset$

  and by induction hypothesis on sw,

  $I = J$ and $sw \in W_I$, thus $w \in W_I = W_J$ since $s \in J$

∎

I.4.6 Theorem D is a fundamental domain for the action of W on U.

Proof. We show that every W-orbit on U meets D in exactly one point.

So let $p \in U$, $\mathcal{O}(p) = \{ \rho^*_w(p) \mid w \in W \}$ its orbit

- $\mathcal{O}(p) \cap D \neq \emptyset$ by definition of U.
- $|\mathcal{O}(p) \cap D| \leq 1$ because otherwise there are

$$q_1, q_2 \in D, \text{ say } q_1 \in C_I,\ q_2 \in C_J,$$

with $\rho^*_w(q_1) = q_2$ for some $w \in W$.

But then $\rho^*_w(C_I) \cap C_J \neq \emptyset$ and by Lemma I.4.5.

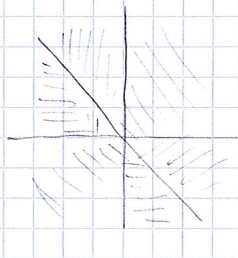
$I = J$, $w \in W_I = W_J$, thus $q_1 = \rho^*_w(q_1) = q_2$ $\square$

I.4.7. Definition Write $w(p)$, wC_I for $\rho^*_w(p)$, $\rho^*_w(C_I)$, for short.

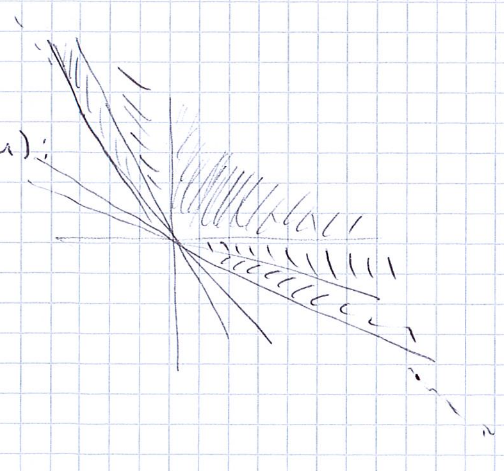
(i) $\Pi_{(W,S)} := \{ w\,\overline{C_I} \mid w \in W,\ I \subseteq S \}$, a polyhedral complex in $V^* \cong \mathbb{R}^{|S|}$

The relation "contained in the boundary of" defines a partial order on this.

A_2:



$I_2(m)$:



(ii) $\Sigma_{(W,S)} := \{ w W_I \mid w \in W, I \subseteq S \}$ is the set of "cosets of standard parabolics"

↳ subgroups of the form W_I, $I \subseteq S$.

The relation of containment defines a partial order.

I.4.8. Theorem There is an isomorphism of partially ordered sets

$$(\Gamma_{(W,S)}, \subseteq) \simeq (\Sigma_{(W,S)}, \subseteq)^{op}$$

Proof For $w_1, w_2 \in W$, $I_1, I_2 \in S$, we have to show

$$w_1 \overline{C_{I_1}} \subseteq w_2 \overline{C_{I_2}} \iff w_1 W_{I_1} \supseteq w_2 W_{I_2}$$

Note first that by definition, for $i = 1,2$:

$$\overline{C_{I_i}} = \bigcup_{J \supseteq I} \overline{C_J} = \bigcup_{J \supseteq I} C_J$$

For the implication "$\Rightarrow$":

$$w_1 \overline{C_{I_1}} \subseteq w_2 \overline{C_{I_2}} \Rightarrow w_2^{-1} w_1 \overline{C_{I_1}} \subseteq \overline{C_{I_2}}$$

$$\Rightarrow w_2^{-1} w_1 C_{I_1} \cap C_J \neq \emptyset \text{ for some } J \supseteq I_2$$

$$\xRightarrow{\text{Lemma I.4.5.}} I_1 \supseteq I_2, \; w_2^{-1} w_1 \in W_{I_1} = W_J \supseteq W_{I_2}$$

$$\Rightarrow w_1 W_{I_1} \supseteq w_2 W_{I_2}.$$

For the implication "$\Leftarrow$":

$$w_1 W_{I_1} \supseteq w_2 W_{I_2} \Rightarrow w_1^{-1} w_2 W_{I_2} \subseteq W_{I_1} \Rightarrow \tilde{w} := w_1^{-1} w_2 \in W_{I_1}$$

Consider now $\tilde{w} C_{I_2}$

$$\operatorname{stab}(\tilde{w} C_{I_2}) = \tilde{w} \operatorname{stab}(C_{I_2}) \tilde{w}^{-1} \overset{\text{Corollary I.4.5.}}{=} \tilde{w} W_{I_2} \tilde{w}^{-1} \subseteq \underbrace{W_{I_1} \tilde{w}^{-1}}_{= W_{I_1}}$$

since $\tilde{w} \in W_{I_1}$

Corollary I.4.5. implies then

$$\overline{\tilde{w} \, C_{I_2}} \supseteq \overline{C_{I_1}}$$

thus $w_2 \overline{C_{I_2}} \supseteq w_1 \overline{C_{I_1}}$ as required $\boxed{4}$