II - A LANGUAGE-THEORETICAL CHARACTERIZATION

II.1. A permutation representation

Let (W,S) be a Coxeter system, let

$$T := \{ wsw^{-1} \mid s \in S,\ w \in W \}$$

called the "set of <u>reflections</u>"

Immediately: $S \subseteq T$, and $t^2 = id$ for all $t \in T$.

Elements of S are called "simple reflections".

Given $s_1, s_2, \ldots, s_k \in S$ we'll let

$$s_1 s_2 \cdots s_k$$

denote both: (1) their product as generators of W

(2) the <u>word</u> formed by listing them in this order (i.e. element of the <u>free monoid</u> / "language" S^*)

II.1.1. <u>Definition</u> Given a word $s_1 s_2 \cdots s_k \in S^*$, define

- $t_i := \underbrace{s_1 s_2 \cdots s_{i-1}}\, s_i\, \underbrace{s_{i-1} \cdots s_2 s_1}$ for $1 \le i \le k$
- $\hat{T}(s_1 \cdots s_k) := (t_1, t_2, \ldots, t_k)$, ordered k-tuple.

} "language" defs

Example. $\hat{T}(abcd) = (a, aba, abcba, abcdcba)$

With respect to the group structure in W, we see

- $t_i \in T$ for all $i = 1 \dots k$
- $t_i s_1 \cdots s_k = s_1 s_2 \cdots s_{i-1} s_i s_{i-1} \cdots s_2 s_1 s_1 s_2 \cdots s_k$
  $= s_1 s_2 \cdots s_{i-1} \hat{s}_i s_{i+1} \cdots s_k$
  ↳ s_i omitted.
- $s_1 \cdots s_i = t_i t_{i-1} \cdots t_1$

For $w \in W$ we define the <u>length</u> of w as

$$l(w) := \min \{ k \in \mathbb{N} \mid w = s_1 \cdots s_k \text{ in } W \text{ for some } s_1 \cdots s_k \in S \}$$

A <u>reduced word</u> is any $s_1 \cdots s_k \in S^*$ such that

$k = l(s_1 \cdots s_k)$ as an element von W.

The word $s_1 \cdots s_k$ is then a <u>reduced expression</u> for $w = s_1 \cdots s_k$.

<u>II.1.2 Lemma</u> If $s_1 \cdots s_k$ is a reduced word, then

$$t_i \neq t_j \quad \text{for all } 1 \leq i < j \leq k$$

<u>Proof</u>: By contraposition.

If $t_i = t_j$ for some $i < j$, then

$$w = \underbrace{t_i t_j}_{= t_i^2 = id} s_1 \cdots s_k = t_i s_1 \cdots \hat{s}_j \cdots s_k =$$
$$= s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k \text{, an}$$

expression shorter than $s_1 \cdots s_k$ for w, thus $s_1 \cdots s_k$ was not reduced □

II-1.3. Definition

(a) For $s_1 \cdots s_k \in S^*$ and $t \in T$ let

$n(s_1 \cdots s_k; t) :=$ the number of times t appears in the tuple $\hat{T}(s_1 \cdots s_k)$

(b) For $s \in S$ and $t \in T$ let

$$\eta(s;t) := \begin{cases} -1 & \text{if } s = t \\ +1 & \text{if } s \neq t \end{cases}$$

We notice immediately the relation among the two:

we have
$$(-1)^{n(s_1 \cdots s_k; t)} = \prod_{i=1}^{k} \underbrace{\eta(s_i; s_{i-1} \cdots s_1 t s_1 \cdots s_{i-1})}_{= (-1) \text{ iff } s_i = s_{i-1} \cdots s_1 t s_1 \cdots s_{i-1} \iff \underbrace{s_1 \cdots s_{i-1} s_i s_{i-1} \cdots s_1}_{i\text{-th entry of } \hat{T}(s_1 \cdots s_k)} = t}$$

(c) Define the set

$$R := T \times \{+1, -1\}$$

and consider the group $\mathrm{Perm}(R)$ of permutations of R.

(d) For $s \in S$ define
$$\pi_s : R \longrightarrow R$$
$$(t, \varepsilon) \longmapsto (sts; \varepsilon\,\eta(s;t))$$

We have: $\underline{\pi_s \in \mathrm{Perm}(R)}$

Proof: π_s is bijective, since

$$\pi_s^2(t, \varepsilon) = \pi_s(sts, \varepsilon\eta(s;t)) = (sstss, \varepsilon\eta(s;t)^2) = (t, \varepsilon)$$

shows $\pi_s^{-1} = \pi_s$ ☐

II.1.4 Theorem ("permutation representation")

(a) The assignment $s \mapsto \pi_s$ extends uniquely to an injective group homomorphism $W \to \mathrm{Perm}(\mathcal{R})$

(b) For all $t \in T$, $\pi_t(t, \varepsilon) = (t, -\varepsilon)$

Proof For (a) we use the universality property. To this end:

- Claim 1: For $s, s' \in S$ with $m := m(s,s') < \infty$,

$$(\pi_s \pi_{s'})^m = id$$

Proof: For clarity, let

$$s_i := \begin{cases} s & \text{if } i \text{ odd} \\ s' & \text{if } i \text{ even} \end{cases} \quad \text{for } 1 \leq i \leq 2m$$

so that $s_1 \cdots s_{2m} = \underbrace{s s' \cdots s s'}_{2m}$. Then, for any $(t, \varepsilon) \in \mathcal{R}$ we have:

$$(\pi_s \pi_{s'})^m (t, \varepsilon) = \pi_{s_1 \cdots s_{2m}} (t, \varepsilon)$$

$$= \Big(\underbrace{\underbrace{s_{2m} \cdots s_1}_{id}\, t\, \underbrace{s_1 \cdots s_{2m}}_{id}}_{(ss')^m}, \underbrace{\varepsilon \prod_{i=1}^{2m} \eta(s_i; s_{i-1} \cdots s_1 t s_1 \cdots s_{i-1})}_{\overset{\text{II.1.3}}{=} \varepsilon\, (-1)^{n(s_1 \cdots s_{2m}; t)}} \Big)$$

$$= \left(t, \varepsilon\, (-1)^{n(s_1 \cdots s_{2m}; t)} \right)$$

But $\hat{T}(s_1 \cdots s_{2m}) = (t_1, t_2, \ldots, t_{2m})$ with

$$t_i = s_1 \cdots s_i s_{i-1} \cdots s_1 = (ss')^{i-1} s',$$

hence $(ss')^m = id$ implies $t_{i+m} = t_i \ \forall i = 1 \ldots m$, and $n(s_1 \cdots s_{2m}; t)$ is <u>even</u>,

thus $(\pi_s \pi_{s'})^m = id$ ┘

- By the universality property we thus have the desired extension to a homom. $w \mapsto \pi_w$, $W \to \mathrm{Perm}(R)$.
- Claim 2: For $w \in W$ and any reduced expr. $w = s_1 \cdots s_k$,

$$(-1)^{n(s_1 \cdots s_k; t)} \quad \text{depends only on } w \text{ and } t$$

Proof: Compute, for any ε,

$$\begin{aligned} \pi_w(t,\varepsilon) &= \pi_{s_k} \cdots \pi_{s_1}(t,\varepsilon) \\ &= \Big(s_k \cdots s_1 t s_1 \cdots s_k,\ \varepsilon \prod_{i=1}^{k} \eta(s_i; s_{i-1} \cdots s_1 t s_1 \cdots s_{i-1})\Big) \\ &= \Big(w t w^{-1},\ \varepsilon (-1)^{n(s_1 \cdots s_k; t)}\Big) \end{aligned}$$

and the claim follows. ┘

- The homomorphism defined above is injective.

Proof: We prove triviality of kernel, i.e., take $w \neq e$ and choose red. expr. $w = s_k \cdots s_1$.

Write $\hat{T}(s_1 \cdots s_k) = (t_1 \cdots t_k)$.

By Lemma II.1.2., all distinct, thus $n(s_1 \cdots s_k; t_i) = 1 \quad \forall i = 1 \dots k$.

We then compute

$$\pi_w(t_i, \varepsilon) = \big(w t_i w^{-1},\ \varepsilon \cdot (-1)\big) \quad \text{for } i = 1 \dots k,$$

in particular $\pi_w \neq \mathrm{id}$ ┘

- Part (b) holds

Proof: Let $t = s_1 s_2 \cdots s_p s_{p-1} \cdots s_1$

  - if $p = 1$, $t = s_1$ and $\pi_{s_1}(s_1, \varepsilon) = (s_1, -1)$ clearly.
  - Let $p > 1$ and apply induction:

$$\pi_t(t,\varepsilon) = \pi_{s_1 \cdots s_p \cdots s_1}(s_1 \cdots s_p \cdots s_1, \varepsilon)$$

$$= \pi_{s_1}\underbrace{\pi_{s_2 \cdots s_p \cdots s_2}(s_2 \cdots s_p \cdots s_2, \varepsilon\eta(s_1; s_2 \cdots s_p \cdots s_2))}_{\pi_{s_1}(s_1 \cdots s_p \cdots s_1, \varepsilon)}$$

(I.V.) $$\overset{\downarrow}{=} \pi_{s_1}(s_2 \cdots s_p \cdots s_2, -\varepsilon\eta(s_1; s_2 \cdots s_p \cdots s_2))$$

$$= (s_1 \cdots s_p \cdots s_1, -\varepsilon\eta^2(\text{———}))$$

$$= (t, -\varepsilon)$$

$\lrcorner$

II.1.5 Scholium

By Claim 2 we can define

$$\eta(w;t) := (-1)^{n(s_1 \cdots s_k; t)}$$

where $s_1 \cdots s_k$ arbitrary red.-expr. for $w \in W$, and $t \in T$.

In particular the proof of Claim 2 gives

$$\pi_w(t,\varepsilon) = (wtw^{-1}, \varepsilon\eta(w^{-1};t))$$

II.2. Properties of reduced words

Let W be any group with a set S of generators of order 2. The definitions of reduced expression, length and "T" still make sense. We then say that

(W,S) satisfies the (strong) [weak] "exchange property" if

for any reduced expression $w = s_1 \cdots s_k$ and any $t \in T$ [$s \in S$],

$\ell(tw) \le \ell(w)$ [$\ell(sw)$] implies tw [sw] $= s_1 \cdots \hat{s}_i \cdots s_k$ for some $i \in [k]$

Remark: Strong $\Rightarrow$ weak.

(W,S) satisfies the "deletion property" if

whenever $w = s_1 \cdots s_k$ and $\ell(w) < k$, then

$$w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k \quad \text{for some } 1 \le i < j \le k$$

II.2.0 Theorem

(W,S) satisfies weak exchange property $\iff$ (W,S) satisfies deletion property

Proof

"$\Rightarrow$" Let $w = s_1 \cdots s_k$, $k > \ell(w)$ and let i be maximal such that $s_i s_{i+1} \cdots s_k$ is not reduced.

Then

$$\ell(s_i s_{i+1} \cdots s_k) \leq \ell(s_{i+1} \cdots s_k)$$

and by "Exchange", we have

$$s_i s_{i+1} \cdots s_k = s_{i+1} \cdots \hat{s}_j \cdots s_k \quad \text{for } j > i$$

multiplying to the left with $s_1 \cdots s_{i-1}$ yields

$$w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$$

☐

"$\Leftarrow$" Consider $w = s_1 \cdots s_k$ reduced, $s \in S$, suppose

$$\ell(s s_1 \cdots s_k) \leq \ell(s_1 \cdots s_k) = k$$

By Deletion, we can delete two letters from $s s_1 \cdots s_k$ obtaining a new expression for sw.

- If s is not one of the two letters:

$$s s_1 \cdots s_k = s s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k \quad \text{, and}$$

$$\ell(w) = \ell(s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k) < k \quad \text{↯}$$

Thus s is one of the deleted letters:

$$sw = s s_1 \cdots s_k = s_1 \cdots \hat{s}_i \cdots s_k \quad \text{as desired}$$

☐

Let us from now again assume that (W,S) is a Coxeter system. The goal of the section is to show that, then, both properties are satisfied.

II.2.1 Lemma The map $\varepsilon: S \to \{+1,-1\}$, $s \mapsto -1$, extends to a group homomorphism $\varepsilon: W \to \{+1,-1\} \simeq \mu_2$

Proof: immediate from UP ☒

II.2.2. Proposition Let (W,S) be a Coxeter system, $u,w \in W$.

(a) $\varepsilon(w) = (-1)^{\ell(w)}$

Pf Since ε is group homomorphism,
$\varepsilon(s_1 \cdots s_k) = (-1)^k = (-1)^{\ell(w)}$ if $s_1 \cdots s_k$ red. expr. for w

(b) $\ell(uw) \equiv \ell(u) + \ell(w) \pmod 2$

Pf from (a) directly $(-1)^{\ell(uw)} = \varepsilon(uw) = \varepsilon(u)\varepsilon(w) \cdots$

(c) $\ell(sw) = \ell(w) \pm 1$ for all $s \in S$

Pf again, because ε is group homom

(d) $\ell(w^{-1}) = \ell(w)$ Pf: $w = s_1 \cdots s_k \Leftrightarrow w^{-1} = s_k \cdots s_1$

(e) $|\ell(u) - \ell(w)| \underset{(i)}{\leq} \ell(uw) \underset{(ii)}{\leq} \ell(u) + \ell(w)$

Pf: (i) Equality when u^{-1} prefix of w or w^{-1} suffix of u
(ii) Clear

(f) $\ell(uw^{-1})$ is a metric on W

Pf: $\ell(uw^{-1}) \geq 0$: clear ; $\ell(uw^{-1}) = 0 \Leftrightarrow u = w$: clear.

symmetry: $\ell(uw^{-1}) = \ell((wu^{-1})^{-1}) \overset{(d)}{=} \ell(wu^{-1})$

triangle ineq: $\ell(uw^{-1}) + \ell(wv^{-1}) \underset{(e)}{\geq} \ell(uv^{-1})$

II.2.3 Lemma For $w \in W$ and $t \in T$,

$$\ell(tw) < \ell(w) \iff \eta(w,t) = -1$$

Proof

"$\Leftarrow$" assume $\eta(w,t) = -1$, choose red. expr. $w = s_1 \cdots s_k$.

Then $n(s_1 \cdots s_k, t)$ is odd, and thus there is some i, $1 \leq i \leq d$, with

$$t = s_1 s_2 \cdots s_i s_{i-1} \cdots s_1$$

So, $\ell(tw) = \ell(s_1 s_2 \cdots s_{i-1} s_{i+1} \cdots s_k) < k = \ell(w)$ ⌟

"$\Rightarrow$" If $\eta(w,t) = 1$, with Scholium II.1.5,

$$\pi_{(tw)^{-1}}(t, \varepsilon) = \pi_{w^{-1}} \pi_t (t, \varepsilon) = \pi_{w^{-1}}(t, -\varepsilon)$$

$$\| \qquad = (w^{-1}tw, -\varepsilon\,\eta(w;t))$$

$$(\cdots, \varepsilon\,\eta(tw;t)) \qquad = (w^{-1}tw, -\varepsilon)$$

in particular, $\eta(tw, t) = -1$ and with "$\Leftarrow$" above

$\ell(ttw) < \ell(tw)$, i.e. $\ell(w) < \ell(tw)$ ∎

II.2.4 Theorem Any Coxeter system (W,S) satisfies the strong exchange property.

Proof: Let $s_1 \cdots s_k$ be a reduced expr. for $w \in W$, let $t \in T$.

If $\ell(tw) < \ell(w)$, by the Lemma $\eta(w,t) = (-1)^{n(s_1 \cdots s_k, t)}$ equals -1, thus $n(s_1 \cdots s_k, t)$ is odd, and so $t = s_1 \cdots s_i s_{i-1} \cdots s_1$ for some $1 \leq i \leq k$.

Thus, $tw = s_1 \cdots s_i s_{i-1} \cdots s_1 s_1 \cdots s_k$

$$= s_1 \cdots \widehat{s_i} \cdots s_k$$

∎

II.2.5. Corollary Let $w = s_1 \cdots s_k$ be a red. expr., and let $t \in T$. The following are equivalent:

(a) $\ell(tw) < \ell(w)$

(b) $tw = s_1 \cdots \hat{s}_i \cdots s_k$ for a unique $1 \le i \le k$

(c) $t = s_1 s_2 \cdots s_i s_{i-1} \cdots s_1$ 〃 〃 〃

Proof (b) $\Longleftrightarrow$ (c) is an easy computation

(a) $\Rightarrow$ (b) is Theorem II.2.4., uniqueness by Lemma II.1.2

(b) $\Rightarrow$ (a) is trivial

∎ 1.4.8?

II.2.6 Definition For $w \in W$ define

$$T_L(w) := \{ t \in T \mid \ell(tw) < \ell(w) \}$$

$$T_R(w) := \{ t \in T \mid \ell(wt) < \ell(w) \}$$

Notice that $T_R(w) = T_L(w^{-1})$.

II.2.7 Corollary $|T_L(w)| = \ell(w)$

(and similarly for T_R)

Proof: Choose a red. expr. $w = s_1 \cdots s_k$

By Corollary II.2.5.,

$$T_L(w) = \{ s_1 \cdots s_i \cdots s_1 \mid 1 \le i \le k \}$$

and all the expressions $s_1 \cdots s_i \cdots s_1$ are distinct by Lemma II.1.2.

∎

II.2.8 Theorem Let W be a group, S a system of generators of order 2. The following are equivalent:

(a) (W,S) is a Coxeter system

(b) (W,S) satisfies the weak Exchange property

(b') (W,S) " " strong " "

(c) (W,S) satisfies the deletion property

Proof: (b) $\Leftrightarrow$ (c) by Proposition II.2.0,

(a) $\Rightarrow$ (b') by Theorem II.2.4, (b') $\Rightarrow$ (b) trivially.

For (b) $\Rightarrow$ (a) we need one last bit of work.

Let (W,S) be a system with the weak exchange property

Define for all $s,s'\in S$ a number

$$m(s,s') = \begin{cases} \min\{n\in\mathbb{N} \mid (ss')^n = id\} & \text{if this exists} \\ \infty & \text{otherwise.} \end{cases}$$

Consider now any relation

$$R: \quad s_1 \cdots s_r = id \quad \text{in } W.$$

We must prove that this is a consequence of the "basic relations": $(ss')^{m(s,s')} = id$ if $m(s,s')<\infty$ (which hold by definition of $m(s,s')$)

- First notice that r must be even, say $r=2k$, because the deletion property holds (it is implied by (b)). We can rewrite

$$R: \quad s_1 \cdots s_k = s'_1 \cdots s'_k$$

- We now prove by <u>induction on k</u> that R can be derived from the basic relations. The case $k=1$ is trivial. Let then $k>1$ and assume that all relations of length less than $2k$ "are O.K." <u>Meaning</u>: can be derived from the basic relations.

<u>Case 1</u>: $s_1 \cdots s_k$ is not reduced.

Let i such that $s_{i+1} \cdots s_k$ is reduced but $s_i \cdots s_k$ not.

By the <u>Exchange property</u>,

$$s_{i+1} \cdots s_k = s_i s_{i+1} \cdots \hat{s}_j \cdots s_k \quad \text{for } j>i.$$

and this equality ("relation") can be generated by basic relations ("is O.K.") because it has length less than $2k$.

If we substitute it into R:

$$R: \; s_1 \cdots s_k = s_1' \cdots s_k'$$

$\Updownarrow$ ← by O.K. moves

$$\underbrace{s_1 \cdots \cancel{s_i} \cancel{s_i} s_{i+1} \cdots \hat{s}_j \cdots s_k = s_1' \cdots s_k'}_{\substack{\text{O.K. by induction.} \\ (\text{length} < 2k)}}$$

⊣

<u>Case 2</u>: $s_1 \cdots s_k$ is reduced.

WLOG assume $s_1 \neq s_1'$, otherwise by $(s_1 s_1)^2 = e$ we reduce to a relation of length $< 2k$.

The Exchange property then gives

$$s_1' s_1 \cdots s_k = s_1 \cdots \hat{s}_i \cdots s_k \quad \text{, thus}$$

$$R' \quad s_1' s_1 \cdots s_i = s_1 \cdots s_{i-1} \quad \text{for some } i \leq k.$$

With R and R' we also know

R'' $\quad s_1 \cdots \hat{s}_i \cdots s_n \quad \left(\overset{R'}{=} s_1' s_1 \cdots s_n\right) \overset{R}{=} \cancel{s_1' s_1'}\, s_2' \cdots s_n'$

which is O.K. because of length $< 2h$, and thus

R''' $\quad s_1' s_1 \cdots \hat{s}_i \cdots s_n = s_1' s_2' \cdots s_n'$ $\quad$ is <u>O.K.</u>

because it is obtained from R'' by a basic relation.

<u>If $i < k$</u> we have that R' is O.K. (length $< 2h$),

and we are done because substituting R' (O.K.) into R''' (O.K.) gives R:

R''' $\quad s_1' \underbrace{s_1 \cdots \hat{s}_i}_{} \cdots s_n = s_1' s_2' \cdots s_n'$

$\downarrow R'$ (O.K.)

$\underbrace{s_1' s_1'}_{(\text{O.K.}) = id} s_1 \cdots s_i \cdots s_n = s_1' \cdots s_n'$

<u>If $i = k$</u> we have R''': $s_1' s_1 \cdots s_{n-1} = s_1' \cdots s_n'$, O.K. (by above)

and it suffices to prove that l.h.s. equals $s_1 \cdots s_n$.

We thus have to show that

R'''' $: \quad s_1' s_1 \cdots s_{n-1} = s_1 \cdots s_n$ $\quad$ is O.K.

<u>But</u> (!) Case 2 applies to R'''' and our procedure either says ~~it~~ it is fine, or leads us to check whether

$(R'''')''''$ $\quad s_1 s_1' s_1 \cdots s_{n-2} = s_1' s_1 \cdots s_{n-1}$ $\quad$ is O.K,

and so on iterating whether

$((R'''')'''')''''$ $\quad s_1' s_1 s_1' s_1 \cdots s_{n-3} = s_1 s_1' s_1 \cdots s_{n-2}$

and ultimately whether

$((R'''')^{\cdots\cdots})''''$ $\quad s_1 s_1' s_1 s_1' \cdots = s_1' s_1 s_1' \cdots$, which is trivially implied by $(s_i s_i')^{m(s_i, s_i')} = id$

$\boxtimes$