CHAPTER III

The weak order and posets of regions

III.1. Partially ordered sets

III.1.1. Definitions. A partially ordered set ("poset") is a pair $(P, \leq)$ where P is a set and $\leq$ a binary relation on P satisfying

- For all $x \in P$: $p \leq p$ (reflexivity)
- For all $x, y \in P$: $x \leq y$ and $y \leq x$ imply $x = y$ (antisymmetry)
- For all $x, y, z \in P$: $x \leq y$ and $y \leq z$ imply $x \leq z$ (transitivity).

We often just refer to "P" if the partial order relationis understood.

We write

$$P_{\leq x} := \{y \in P \mid y \leq x\} \qquad P_{\geq x} := \{y \in P \mid y \geq x\}$$

If $x \leq y$ in P, t The *interval* between x and y is the set

$$[x, y] := P_{\geq x} \cap P_{\leq y}.$$

We say that y *covers* x if $[x, y] = \{x, y\}$. The *Hasse diagram* of a poset P is a drawing of the graph with P as set of vertices, and with an edge xy whenever y covers x.

III.1.2. Examples. For $n \in \mathbb{N}_{>0}$ let

(i) $[n]$: The set $\{1, 2, \ldots, n\}$ with the natural order.
(ii) $B_n := (\mathscr{P}([n]), \subseteq)$, the *boolean poset* of all subsets of $\{1, \ldots, n\}$, ordered by inclusion.

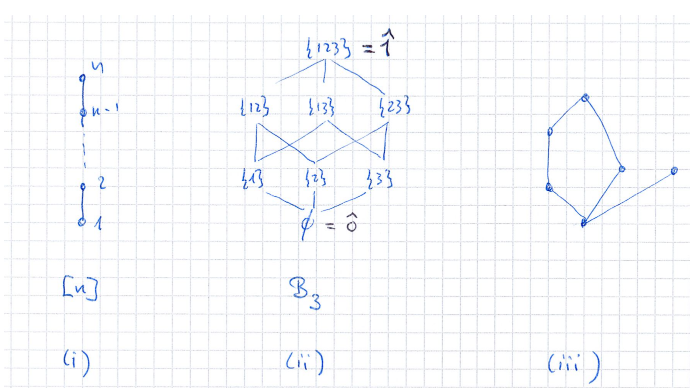


FIGURE 1

III.1.3. Definitions. The poset P is *bounded below* if it possesses a unique minimal element, i.e., if there is an element $\hat{0} \in P$ with $P = P_{\geq \hat{0}}$. Accordingly, P is *bounded below* if it possesses a unique maximal element, i.e., if there is an element $\hat{1} \in P$ with $P = P_{\leq \hat{1}}$. P is *bounded* if it is both bounded below and bounded above.

If P is bounded below, we call *atoms* of P the elements covering $\hat{0}$.

Given posets $(P, \leq_P)$, $(Q, \leq_Q)$, a function $f : P \to Q$ is *order preserving* if, for every $x, y \in P$,

$$x \leq_P y \Rightarrow f(x) \leq_Q f(y).$$

We call f an *isomorphism of posets* if it has an order-preserving inverse (notice: if P is finite, it is enough to show that f is order preserving and bijective).

III.1.4. Definitions. let $x, y \in P$. The *meet* $x \wedge y$ and the *join* $x \vee y$, are defined by

$$P_{\leq x \wedge y} = P_{\leq x} \cap P_{\leq y}, \qquad P_{\geq x \vee y} = P_{\geq x} \cap P_{\geq y}.$$

P is called a *meet semilattice* (resp. join semilattice) if $x \wedge y$ (resp. $x \vee y$) exists for all $x, y \in P$. If P is both a meet- and a join- semilattice, then P is called a *lattice*.

The *rank* of a poset P, denoted by $\mathrm{rk}(P)$, is the length of the longest chain in P. For every element $x \in P$ let $\mathrm{rk}(x) := \mathrm{rk}[\hat{0}, x]$.

III.1.5. Lemma. *Let P be a bounded poset such that for every $x, y \in P$ the join $x \vee y$ exists. Then P is a lattice.*

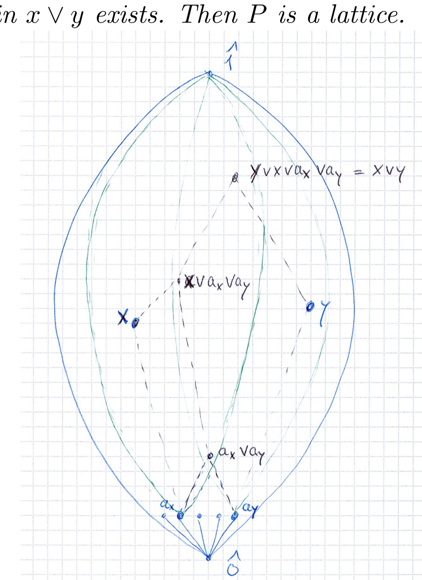
PROOF. We must prove that for all $x, y \in P$ the meet $x \wedge y$ exists. To this end, choose $x, y \in P$ and consider

$$P_{\leq x} \cap P_{\leq y}.$$

Since P is bounded, this set is not empty. Since joins exist, this set must have a unique maximal element: this element satisfies the definition of $x \wedge y$. □

III.1.6. Lemma. *Let P be a bounded poset of finite rank such that, for any $x, y, z \in P$, if both x, y cover z then the join $x \vee y$ exists. Then P is a lattice.*

PROOF. By Lemma III.1.5 it is enough to prove that $x \vee y$ exists for every $x, y \in P$. This we prove by induction on $\mathrm{rk}(P)$, the claim being clearly true if $\mathrm{rk}(P) \leq 2$.
Assume then $\mathrm{rk}(P) = k > 2$ and let $x, y \in P$ that do not cover a common element. Let $a_1, a_2 \in P$ be two elements covering $\hat{0}$ such that $a_1 \leq x$, $a_2 \leq y$. If $a_1 = a_2$, then $x, y \in P_{\geq a_1}$ and by induction $x \vee y$ exists. Otherwise, simce both a_1, a_2 cover $\hat{0}$ we know that $a_1 \vee a_2$ exists. Now, $x \vee a_1 \vee a_2$ exists in $[a_1, \hat{1}]$, $y \vee x \vee a_1 \vee a_2$ exists in $[a_2, \hat{1}]$. The fact that $a_1 \leq x$ and $a_2 \leq y$ implies $x \vee y \vee a_1 \vee a_2 = x \vee y$. □



III.2. Weak order of Coxeter groups

III.2.1. Definition. *Let (W, S) be a Coxeter system and let $u, w \in W$. We say:*

(i) $u \leq_R w$ *if there are* $s_1 \cdots s_k \in S$ *such that* $w = us_1 \cdots s_k$ *and* $\ell(us_1 \cdots s_i) = \ell(u) + i$ *for all* $i = 1, \ldots, k$.
(ii) $u \leq_L w$ *if there are* $s_1 \cdots s_k \in S$ *such that* $w = s_k s_{k-1} \cdots s_1 u$ *and* $\ell(s_1 s_{1-1} \cdots s_1 u) = \ell(u) + i$ *for all* $i = 1, \ldots, k$.

It is immediate to verify that these define partial orderings on W. The two posets $(W, \leq_R)$ and $(W, \leq_L)$ are isomorphic through the function

$$W \to W, \quad w \mapsto w^{-1},$$

which is easily checked to be an order preserving bijection.

III.2.2. Remark. For all $u, w \in W$,

(i) $u \leq_R w$ *if and only if* $\ell(u) + \ell(u^{-1}w) = \ell(w)$.

Proof. Choose a reduced expression $u = s_1 \cdots s_{\ell(u)}$.
• If $u \leq_R w$, by definition there are $s'_1, \cdots, s'_l$ with $w = us'_1 \cdots s'_l$ (so $\ell(u^{-1}w) \leq l$), and $\ell(w) = \ell(u) + l$. Together,

$$\ell(u) + l = \ell(w) \overset{II.2.2.(e)}{\leq} \ell(u) + \ell(u^{-1}w) \leq \ell(u) + l$$

implying equality throughout, hence the desired equation.
• Conversely, assume $\ell(u) + \ell(u^{-1}w) = \ell(w)$ and choose a reduced expression $u^{-1}w = s'_1 \cdots s'_{\ell(u^{-1}w)}$. Suppose, by way of contradiction, that for some i we have $\ell(us'_1 \cdots s'_i) < \ell(u) + i$. Then we would have

$$\begin{aligned}\ell(w) = \ell(us'_1 \cdots s'_{\ell(u^{-1}w)}) &\overset{II.2.2.(e)}{\leq} \ell(us'_1 \cdots s'_i) + \ell(s'_{i+1} \cdots s'_{\ell(u^{-1}w)}) \\ &< \ell(u) + i + (\ell(u^{-1}w) - i) = \ell(u) + \ell(u^{-1}w),\end{aligned}$$

reaching the desired contradiction. □

(ii) $u \leq_R w$ *if and only if there are reduced expressions*

$$u = s_1 \cdots s_k \qquad \textit{and} \qquad w = s_1 \cdots s_k s'_1 \cdots s'_l.$$

Proof. Analogous (exercise). □

III.2.3. Recall / Definition. Remember from Definition II.2.6. that for all $w \in W$ we have a set

$$T_L(w) := \{t \in T \mid \ell(tw) < \ell(w)\}$$

which, by Corollary II.2.5., equals

$$T_L(w) = \{s_1 \cdots s_i s_{i-1} \cdots s_1 \mid 1 \leq i \leq k\}$$

where $w = s_1 \cdots s_k$ is a reduced expression for w. By Lemma II.1.2. we also have

$$|T_L(w)| = \ell(w).$$

III.2.4. Proposition. *For $u, w \in W$, $u \leq_R w$ if and only if $T_L(u) \subseteq T_L(w)$.*

PROOF. Suppose first $u \leq_R w$. Then, by Remark III.2.2 we can find a reduced expression $w = s_1 \cdots s_l$ with $u = s_1 \cdots s_k$ for some $k \leq l$. As recalled in III.2.3, we then have

$$\begin{aligned} T_L(u) &= \{s_1 \cdots s_i s_{i-1} \cdots s_l \mid 1 \leq i \leq k\} \\ &\subseteq \{s_1 \cdots s_i s_{i-1} \cdots s_l \mid 1 \leq i \leq l\} = T_L(w). \end{aligned}$$

For the reverse direction, suppose $T_L(u) \subseteq T_L(w)$ and choose a reduced expression $u = s_1 \cdots s_k$. By Remark III.2.2 it is enough to find a reduced expression for w of the form

$$w = s_1 \cdots s_k \underbrace{s'_1 \cdots \cdots s'_{l-k}}_{l-k \text{ letters from } S},$$

where $l = \ell(w)$. We will show this by proving inductively

Claim: *For all $i = 0, \ldots, k$ the following statement holds.*
A(i): *there is a reduced expression for w of the form*

$$w = s_1 \cdots s_i \boxed{(l-i) \text{ letters from } S}$$

Proof of the claim. The statement is clearly true for $i = 0$. Suppose that $A(i)$ holds for some $i < k$ and consider $A(i+1)$.
For $j = 1, \ldots, k$ write $t_j := s_1 \cdots s_j s_{j-1} \cdots s_k$. With III.2.3 we know $t_{i+1} \in T_L(u)$, hence

$$t_{i+1} \in T_L(w).$$

With $A(i)$ we can write $w = s_1 \cdots s_i s'_1 \cdots s'_{l-i}$ and, using III.2.3, t_{i+1} must then have the form

$$t_{i+1} = (s_1 \cdots s_i s'_1 \cdots s'_m)(s'_{m-1} \cdots s'_1 s_i \cdots s_1)$$

for some $m \leq l-i$. (In fact, III.2.3 allows also for $t_{i+1} = s_1 \cdots s_j s_{j-1} \cdots s_1 = t_j$ for some $j < i+1$, but we can exclude this possibility because Lemma II.1.2. implies $t_j \neq t_{i+1}$ for all $j \neq i+1$). We can rewrite w as

$$\begin{aligned} w = t_{i+1}(t_{i+1}w) &\overset{II.2.5.(c)}{=} t_{i+1}(s_1 \cdots s_i s'_1 \cdots s'_{m-1} \widehat{s'_m} s'_{m+1} \cdots s'_{l-i}) \\ &= (s_1 \cdots s_{i+1} s_i \cdots s_1) s_1 \cdots s_i s'_1 \cdots s'_{m-1} \widehat{s'_m} s'_{m+1} \cdots s'_{l-i} \\ &= s_1 \cdots s_{i+1} \underbrace{s'_1 \cdots \widehat{s'_m} \cdots s'_{l-i}}_{l-(i+1) \text{ letters}} \end{aligned}$$

as required.

□

III.3. The poset of regions of an arrangement of hyperplanes in $\mathbb{R}^d$

III.3.1. Definitions. Let

$$\mathcal{A} := \{H_1, \ldots, H_n\}$$

denote an arrangement of hyperplanes in $\mathbb{R}^d$, i.e., there are vectors $a_1, \ldots, a_n \in \mathbb{R}^d$ and real numbers $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$H_i = \{x \in \mathbb{R}^d \mid \langle x, a_i \rangle = c_i\}. \tag{1}$$

The arrangement $\mathcal{A}$ is called *central* if all $c_i = 0$ - this means, all hyperplanes pass through the origin of $\mathbb{R}^d$.

Once the a_i are chosen, for each i we can also talk about

$$H_i^+ := \{x \in \mathbb{R}^d \mid \langle x, a_i \rangle > c_i\}, \quad H_i^+ := \{x \in \mathbb{R}^d \mid \langle x, a_i \rangle < c_i\}.$$

A moment's thought reveals that the complement of $\mathcal{A}$,

$$M(\mathcal{A}) := \mathbb{R}^d \setminus \bigcup_{i=1}^{n} H_i,$$

is a disjoint union of nonempty open cones of the form

$$\bigcap_{i=1}^{n} H_i^{\epsilon_i} \qquad \text{for some } \epsilon \in \{+, -\}^n.$$

These open cones are the *chambers*, or *regions* of the arrangement. The cones themselves do not depend on the choice of the a_i, thus we denote by

$$\mathcal{R}(\mathcal{A})$$

the set of chambers of $\mathcal{A}$.

The *separation set* of $C_1, C_2 \in \mathcal{R}(\mathcal{A})$ is

$$\begin{aligned} S(C_1, C_2) &:= \{H_i \mid C_1 \subset H_i^{\epsilon},\ C_2 \subset H_i^{-\epsilon} \text{ for some } \epsilon \in \{+, -\}\} \\ &= \{H_i \mid \text{ given } p_1 \in C_1, p_2 \in C_2, \text{ the segment } \overline{p_1 p_2} \text{ meets } H_i\}. \end{aligned}$$

For every $B \in \mathcal{R}(\mathcal{A})$ define a relation $\preceq_B$ on $\mathcal{R}(\mathcal{A})$ by

$$C_1 \preceq_B C_2 \text{ if and only if } S(B, C_1) \subseteq S(B, C_2).$$

It is easy to see that this relation is a partial order. We will write $[C_1, C_2]_B$ for the interval determined by $\preceq_B$.

The *poset of regions of $\mathcal{A}$ based at B* is

$$\mathcal{R}(\mathcal{A}, B): \quad \mathcal{R}(\mathcal{A}) \text{ partially ordered by } \preceq_B .$$

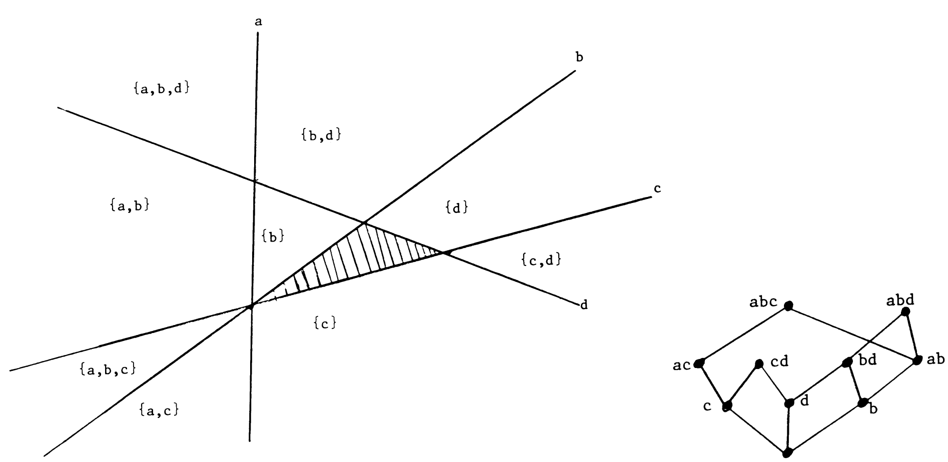


FIGURE 2. An arrangement of hyperplanes (lines) in $\mathbb{R}^2$ and (the Hasse diagram of) its poset of regions based at the shaded region.

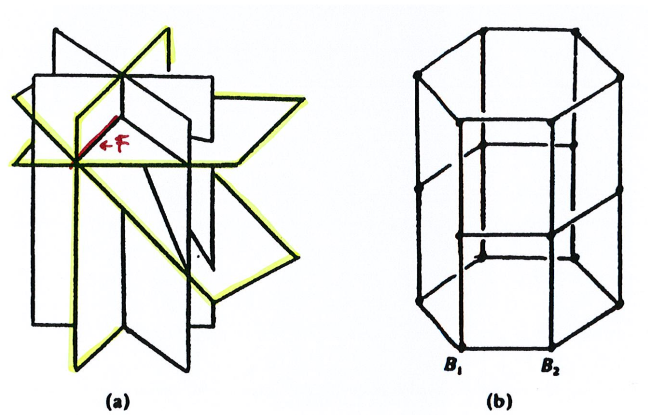


FIGURE 3. A central arrangement and (the Hasse diagram of) its poset of regions.

III.3.2. Remark. $S(B, C) = \{H_i \mid C \subseteq H_i^-\}$.

(i) *$C_1 \preceq_B C_2$ is equivalent to $S(B, C_2) = S(B, C_1) \uplus S(C_1, C_2)$.*

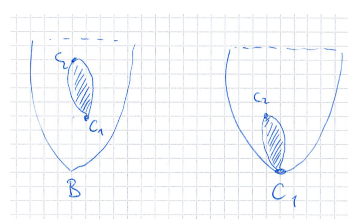
PROOF. The right-to-left implication is trivial. For the other direction notice that there is no loss of generality in choosing a_i such that $B \subseteq H_i^+$ for all i. Then, $S(B, C_2) = \{H_i \mid C_2 \subseteq H_i^-\}$. Now write

$$\begin{aligned} S(B, C_2) &= \{H_i \in S(B, C_2) \mid C_1 \subseteq H_i^-\} \uplus \{H_i \in S(B, C_2) \mid C_1 \subseteq H_i^+\} \\ &= S(B, C_1) \uplus \{H_i \in \mathcal{A} \mid C_2 \subseteq H_i^-,\ C_1 \subseteq H_i^+\} \\ &= S(B, C_1) \uplus S(C_1, C_2). \end{aligned}$$

□

(ii) *The identity map induces an isomorphism $[C_1, C_2]_B \simeq [C_1, C_2]_{C_1}$.*

PROOF. Let $R_1, R_2 \in [C_1, C_2]_B$.
It is enough to prove that



$$R_1 \preceq_B R_2 \Leftrightarrow R_1 \preceq_{C_1} R_2.$$

With (i) we write

$$\begin{aligned} R_1 \preceq_B R_2 \Leftrightarrow S(B, R_2) =& S(B, R_1) \uplus S(R_1, R_2) \\ \overset{C_1 \preceq_B R_1}{=}& S(B, C_1) \uplus S(C_1, R_1) \uplus S(R_1, R_2) \end{aligned}$$

Since $C_1 \preceq_B R_2$ implies $S(B, R_2) = S(B, C_1) \uplus S(C_1, R_2)$, the above is equivalent to

$$S(C_1, R_2) = S(C_1, R_1) \uplus S(R_1, R_2)$$

which, by (i), means $R_1 \preceq_{C_1} R_2$. □

III.3.3. Proposition. Clearly $\mathcal{R}(\mathcal{A}, B)$ always has a unique minimal element, namely B. If $\mathcal{A}$ is central, $\mathcal{R}(\mathcal{A}, B)$ is bounded for every B.

PROOF. Fix $B \in \mathcal{R}(\mathcal{A})$. If $\mathcal{A}$ is central, $-B \in \mathcal{R}(\mathcal{A})$ and clearly $S(B, -B) = \mathcal{A}$. Hence, $C \preceq_B -B$ for all $C \in \mathcal{R}(\mathcal{A})$. □

III.3.4. Definition. Let $B \in \mathcal{R}(\mathcal{A})$. The set of *walls* of B is

$$\mathcal{W}(B) := \{H \in \mathcal{A} \mid \dim H \cap \overline{B} = d - 1\}.$$

The set of *faces* of B is

$$\mathscr{F}(B) := \{\overline{B} \cap X \mid X = \bigcap_{H \in A} H \text{ for some } A \subseteq \mathcal{W}(B)\},$$

it is the set of faces of the closed polytope $\overline{B}$. We write $\mathscr{F}^{(1)}(B)$ for the set of faces of codimension 1. Given any $F \in \mathscr{F}(B)$ let

$$\mathcal{H}(F) := \{H \in \mathcal{A} \mid F \subseteq H\}, \quad \mathcal{W}(B, F) = \mathcal{H}(F) \cap \mathcal{W}(B).$$

Call a chamber $B \in \mathcal{R}(\mathcal{A})$ *simplicial* if the hyperplanes in $\mathcal{W}(B)$ have linearly independend normals. This is equivalent to saying that $\overline{B}$ is a simplicial cone.

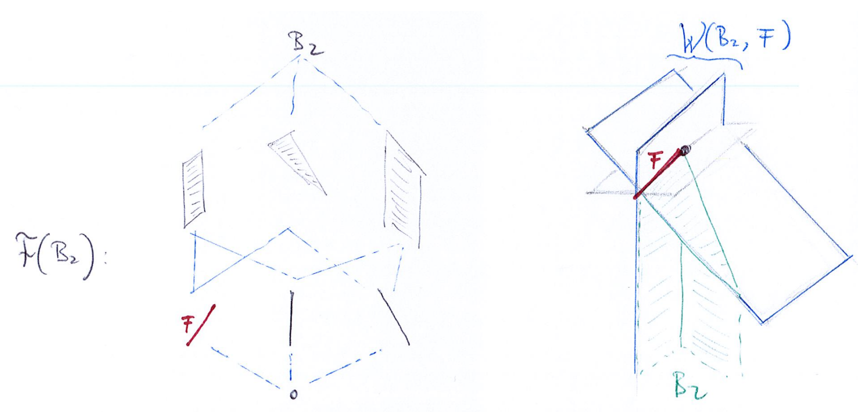


FIGURE 4. The set $\mathscr{F}(B_2)$ (partially ordered by inclusion) and the set $\mathcal{W}(B_2, F)$ for the chamber B_2 and the face F of the arrangement of Figure 3, where the set $\mathcal{H}(F)$ is highlighted in yellow.

III.3.5. Lemma. Let $B \in \mathcal{R}(\mathcal{A})$.

(i) For every $F \in \mathscr{F}(B)$ there is a unique chamber $R(F) \in \mathcal{R}(\mathcal{A})$ such that

$$S(B, R(F)) = \mathcal{H}(F).$$

Moreover, $S(B, C) \supseteq \mathcal{W}(B, F)$ implies $S(B, R(F)) = \mathcal{H}(F)$, and hence $C \succeq_B R(F)$.

(ii) The set of atoms of $\mathcal{R}(\mathcal{A}, B)$ is $\{R(F) \mid F \in \mathscr{F}^{(1)}(B)\}$

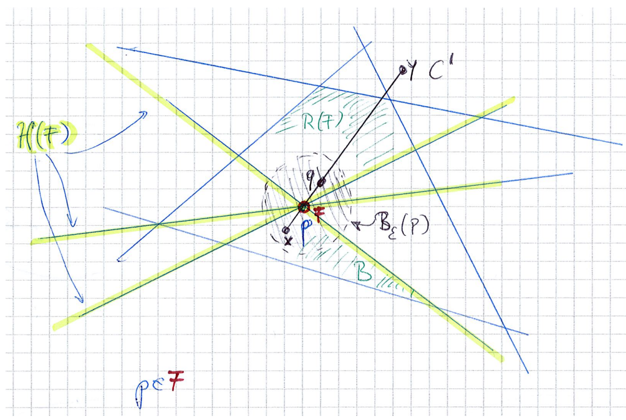
(iii) If B is simplicial, any set $R(F_1), \ldots, R(F_k)$ of atoms of $\mathcal{R}(\mathcal{A}, B)$ has the join

$$R(F_1) \vee \ldots \vee R(F_k) = R(F_1 \cap \ldots \cap F_k).$$

Proof.

(i) Let $p \in \mathrm{relint}(F)$.
First, we claim: there is $\epsilon > 0$ such that $B_\epsilon(p) \cap H = \emptyset$ for every hyperplane $H \notin \mathcal{H}(F)$.
[Proof. p is in the complement, hence in a unique chamber C_p of the arrangement $\mathcal{A} \setminus \mathcal{H}(F)$. Since C_p is open, there is $\epsilon > 0$ with $B_\epsilon(p) \subseteq C_p$. This satisfies the claim.]
Choose then $x \in B \cap B_\epsilon(p)$ (which is nonempty because $p \in \overline{C}$) and consider $q := p - x \in B_\epsilon(p)$. Then q is in the complement of $\mathcal{A}$.
[Proof:
• for $H \in \mathcal{H}(F)$, the line $\overline{px}$ does not lie in H, hence only meets H in x.
• For $H \in \mathcal{A} \setminus \mathcal{H}(F)$, $q \in B_\epsilon(p)$ implies $q \notin H$.]
Thus q is contained in a unique chamber of $\mathcal{A}$, that we call $R(F)$, for which

$$S(B, R(F)) = \mathcal{H}(F).$$



Every other $C' \in \mathcal{R}(\mathcal{A})$ with $S(B, C') \supseteq \mathcal{H}(F)$ must lie in the chamber of $\mathcal{H}(F)$ containing $R(F)$, and for any $y \in C'$ the line $\overline{yp}$ meets a nonempty subset of $\mathcal{A} \setminus \mathcal{H}(F)$ and, continuing it by $< \epsilon$ past p, will further meet only hyperplanes in $\mathcal{H}(F)$. Hence we have a segment from B to C' that meets more than the hyperplanes in $\mathcal{H}(F)$. This proves uniqueness.

Looking closer at the situation, we see that in the central arrangement $\mathcal{H}(F) - p$, the set of walls of the chamber $-(B - p) \ni q - p$ is, by symmetry, again $\mathcal{W}(B, F) - p$. Therefore, in $\mathcal{H}(F)$ requiring $S(B, C) \supseteq \mathcal{W}(B, F)$ implies $S(B, C) \supseteq \mathcal{H}(F)$.

(ii) Clearly, for every $F \in \mathscr{F}^{(1)}(B)$, $R(F)$ is an atom. Conversely, if C is an atom of $\mathcal{R}(\mathcal{A}, B)$ then $|S(B, C)| = 1$. This means that for $x \in B$, $y \in C$, the segment $\overline{xy}$ meets only H, and does so in a point $p \in H \cap \overline{C}$ which is in (thus: possesses an ϵ-ball inside) the complement of $\mathcal{A} \setminus \{H\}$. Therefore $p \in \mathrm{relint}(F)$ for $F = \overline{B} \cap H \in \mathscr{F}^{(1)}(B)$, and by the uniqueness of (i), $C = R(F)$.

(iii) If B is simplicial, then $F := F_1 \cap \ldots \cap F_k$ is a codimension k face with $\mathcal{W}(B, F) = \{H_1, \ldots, H_k\}$, where $F_i = \overline{C} \cap H_i$.
We know from (i) that $R(F) \succeq_B R(F_i)$ for all $i = 1, \ldots, k$.

On the other hand, $C \succeq_B R(F_i)$ for all $i = 1, \ldots, k$ implies $S(B, C) \supseteq \mathcal{W}(B, F)$ and thus by (i) again $C \succeq_B R(F)$.

□

III.3.6. Theorem. *If $\mathcal{A}$ is a simplicial arrangement, then the poset $\mathcal{R}(\mathcal{A}, B)$ is a lattice for all choices of B.*

Proof. Fix an arbitrary $B \in \mathcal{R}(\mathcal{A})$. We will use Lemma III.1.6 in order to show that $\mathcal{R}(\mathcal{A}, B)$ is a lattice.

Let $C_1, C_2, C \in \mathcal{R}(\mathcal{A}, B)$ and suppose that both C_1 and C_2 cover some C.

Since C is simplicial, Lemma III.3.5.(iii) shows that the join $C_1 \vee_C C_2$ exists in $\mathcal{R}(\mathcal{A}, C)$. It is, by definition, containd in $[C, -B]_C$. Via the isomorphism of Remark III.3.2.(ii) between the intervals $[C, -B]_B$ and $[C, -B]_C$, we conclude that $C_1 \vee_C C_2 = C_1 \vee_B C_2$, so the join exists in $\mathcal{R}(\mathcal{A}, B)$ as well. □

III.3.7. Remark. There are examples of non-simplicial arrangements $\mathcal{A}$ with some $B \in \mathcal{R}(\mathcal{A})$ for which $\mathcal{R}(\mathcal{A}, B)$ is a lattice. However, the following characterization of simplicial arrangements holds.

III.3.8. Theorem. A central arrangement $\mathcal{A}$ is simplicial if and only if $\mathcal{R}(\mathcal{A}, B)$ is a lattice for every choice of B.

Proof. Theorem III.3.6 proves one direction. The interested reader can find a proof of the other direction as [**3**, Theorem 3.1]. □

III.4. The weak order of finite Coxeter groups is a lattice

Let (W, S) be a finite Coxeter group. Recall from Chapter I the set of *roots*

$$\Phi_W = \{\rho_w(\alpha_s) \mid w \in W,\, s \in S\} \subseteq V \simeq \mathbb{R}^d$$

Recall the associated *Coxeter arrangement* in V^*:

$$\mathcal{A} = \{\rho_w^*(H_s) \mid s \in S\}$$

To every reflection $t = usu^{-1} \in T$ we can associate a hyperplane

$$H_t = \{p \in V^* \mid \langle p, \rho_u(\alpha_s)\rangle = 0\}.$$

III.4.1. Lemma. $\mathcal{A} = \{H_t \mid t \in T\}$.

Proof. From the definition we see

$$H_t = \{p \in V^* \mid \langle \rho_{u^{-1}}^*(p), \alpha_s\rangle = 0\} = \{\rho_u^*(q) \mid \langle q, \alpha_s\rangle = 0\} = \rho_u^*(H_s)$$

and with an analogous computation $\rho_w^*(H_s) = H_{wsw^{-1}}$, thus the claim holds.

□

III.4.2. Proposition. *Let $w \in W$. Then, for all $t \in T$:*

$$H_t \in S(C, \rho_w^*(C)) \text{ if and only if } t \in T_L(w).$$

Proof. "$\Leftarrow$" By Definition III.2.3, $t \in T_L(w)$ means $\ell(tw) < \ell(w)$. By Corollary II.2.5., this in turn means that, if $w = s_1 \cdots s_k$ is a reduced expression, there is an i with

$$t = \underbrace{s_1 \cdots s_{i-1}}_{=:u} s_i s_{i-1} \cdots s_1.$$

Since $u^{-1} = s_{i-1} \cdots s_1$, we immediately have $\ell(s_i u^{-1}) > \ell(u^{-1})$. By Lemma I.4.4., equivalently $\rho_{u^{-1}}^*(C) \subseteq H_{s_i}^+$ or, otherwise said, for all $p \in C$

$$\langle p, \rho_u(\alpha_{s_i})\rangle = \langle \rho_{u^{-1}}^*(p), \alpha_{s_i}\rangle > 0.$$

Let then $v := s_i \cdots s_k$, so that $w = uv$.

Again, by Lemma I.4.4. $\ell(s_i v) < \ell(v)$ is equivalent to

$$\rho_v^*(C) \subseteq H_{s_i}^-,$$

hence

$$\begin{aligned}\rho_w^*(C) = \rho_u^* \rho_v^*(C) \subseteq \rho_u^*(H_{s_i}^-) &= \{\rho_u^*(q) \mid \langle q, \alpha_{s_i} \rangle < 0\} \\ &= \{\rho_u^*(q) \in V^* \mid \langle \rho_u^*(q), \rho_u(\alpha_{s_i}) \rangle < 0\} \\ &= \{p \in V^* \mid \langle p, \rho_u(\alpha_{s_i}) \rangle < 0\}.\end{aligned}$$

"$\Rightarrow$" For the other implication we proceed by contraposition: suppose then $t \notin T_L(w)$. This means $\ell(tw) > \ell(w)$. Applying the argument carried out above to t and tw we see that

$$H_t \in S(C, \rho_{tw}^*(C)). \tag{2}$$

Now, for every $p \in C$, writing $t = usu^{-1}$ we have

$$\begin{aligned}\langle \rho_{tw}^*(p), \rho_u(\alpha_s) \rangle &= \langle \rho_w^*(p), \rho_{tu}(\alpha_s) \rangle = \langle \rho_w^*(p), \rho_{us}(\alpha_s) \rangle \\ &= \langle \rho_w^*(p), \rho_u(-\alpha_s) \rangle = -\langle \rho_w^*(p), \rho_u(\alpha_s) \rangle\end{aligned}$$

and in particular

$$H_t \in S(\rho_{tw}^*(C), \rho_w^*(C)). \tag{3}$$

Equations (2) and (3) together imply

$$H_t \notin S(C, \rho_w^*(C))$$

as required. □

III.4.3. Theorem. *The function*

$$\phi : W \to \mathcal{R}(\mathcal{A}_W), \quad w \mapsto \rho_w^*(C)$$

defines a poset isomorphism

$$(W, \leq_R) \simeq \mathcal{R}(\mathcal{A}, C)$$

Proof. The function ϕ is a bijection by Chapter I, and is order preserving by Theorem 4.2 and Proposition 2.4 □

III.4.4. Corollary. *The Bruhat order of a finite Coxeter system is a lattice.*

Proof. By Chapter I the arrangement $\mathcal{A}_W$ is simplicial. By Theorem 3.10 then, $\mathcal{R}(\mathcal{A}_W, C)$ is a lattice, thus – via the isomorphism of Theorem 4.3 – so is $(W, \leq_R)$. □

III.5. Sources

Section III.1 includes material from [**4**] and [**3**].

Section III.2 is a selection from [**1**, Chapter 3].

Section III.3 includes material from [**2**] and [**3**].

Figure 2 is taken from [**2**], and Figure 3 is modified from [**3**].

Bibliography

[1] Anders Björner, Francesco Brenti. *Combinatorics of Coxeter groups.* Springer GTM.

[2] Paul Edelman. *A partial order on the regions of* $\mathbb{R}^d$ *dissected by hyperplanes.* Transactions of the American Mathematical Society, Vol. 283, n. 2, 1984, 617–631.

[3] Anders Björner, Paul Edelman, Günter Ziegler. *Arrangements with a lattice of regions.* Discrete and Computational Geometry, Vol. 5, 1990, p. 263–288.

[4] Ricard Stanley. *Enumerative combinatorics, Vol. 1.* Oxford University Press.