Review:

- Matroids : $\rho : E \to \mathbb{N}$ with r0, r1, r2

  Example: • Graph G, edges E, for $A \subseteq E$

  $\rho(A) :=$ card. of max. spanning tree

  • E set of vectors in $\mathbb{K}^d$: $\rho(A) = \dim_{\mathbb{K}} \langle A \rangle$

  is matroid.

- Tutte polynomial of ρ :

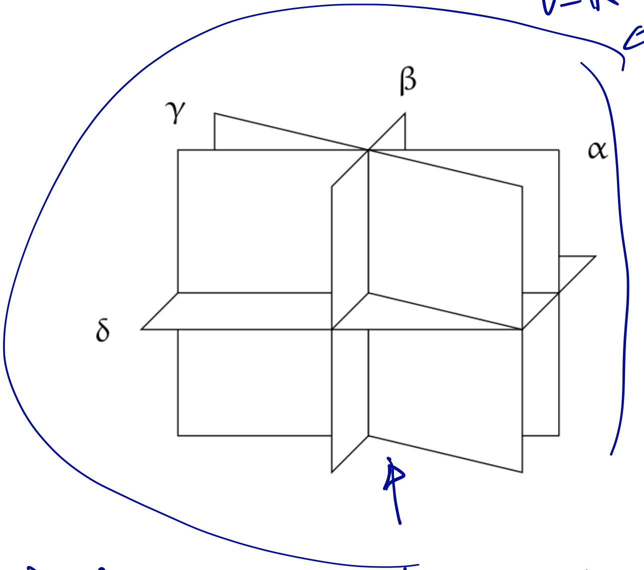
$$\underline{T_\rho(x,y)} = \sum_{A \subseteq E} (x-1)^{\rho(E)-\rho(A)} (y-1)^{|A|-\rho(A)}$$

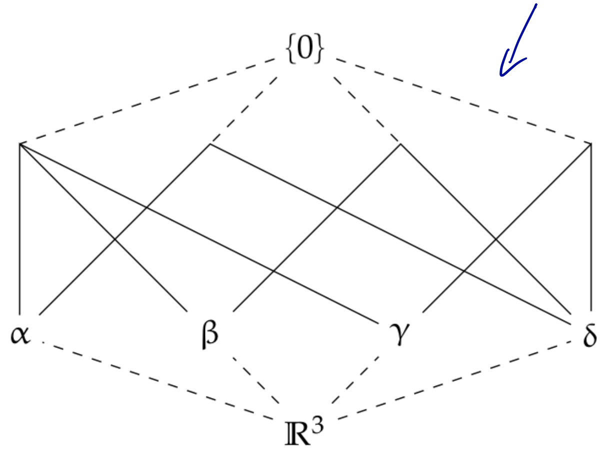
$!\boxed{\exists}!$ "<u>Universality</u>" w.r.t. "being a <u>Tutte-Grothendieck</u> inv."

3. Hyperplane arrangements

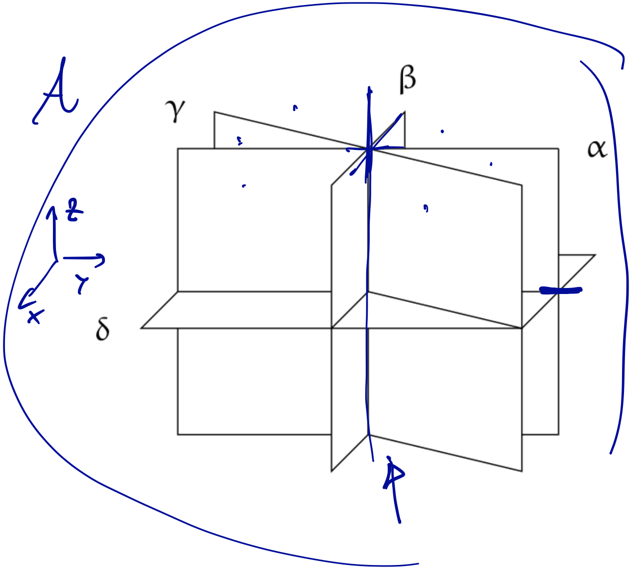
$\mathcal{A} = \{$

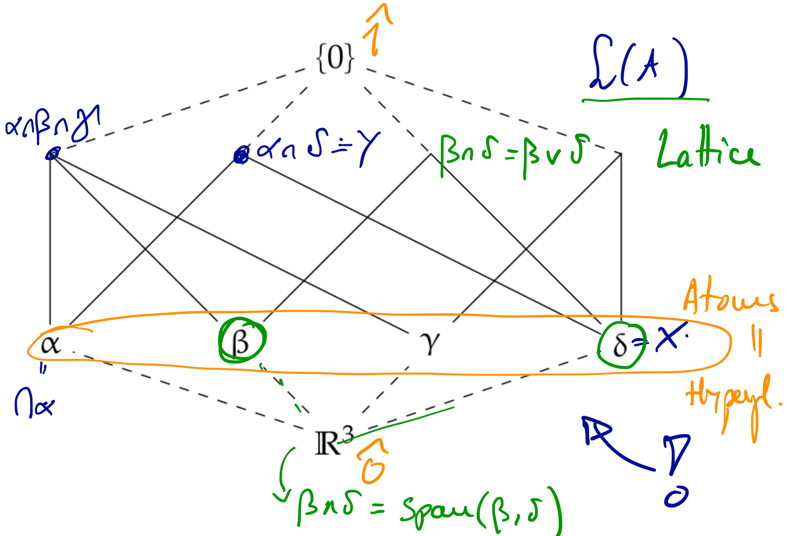
$V = \mathbb{R}^3$





Def: V any vectorspace, dimension $d < \infty$. An <u>arrangement</u> (of Hyps) is any finite set $\mathcal{A} = \{H_1, \dots, H_n\}$ of linear, codimension 1 subspaces of V.





<u>Example</u>: $\mathcal{A} = \{\alpha, \beta, \gamma, \delta\}$

$\alpha: x=0 \qquad \gamma: x=y$

$\beta: y=0 \qquad \delta: z=0 \qquad$ in $\mathbb{R}^3$

<u>Def</u>: $\mathcal{L}(\mathcal{A}) = \left\{ \bigcap_{i\in I} H_i \;\middle|\; I \subseteq [n] \right\}$ partial order: by reverse inclusion. (i.e. $X \leq Y$ if $X \supseteq Y$)

<u>"Project"</u>: See how, from $\mathcal{L}(\mathcal{A})$, we can compute the number of "pieces" in the complement $\mathbb{R}^3 \setminus \bigcup\mathcal{A}$

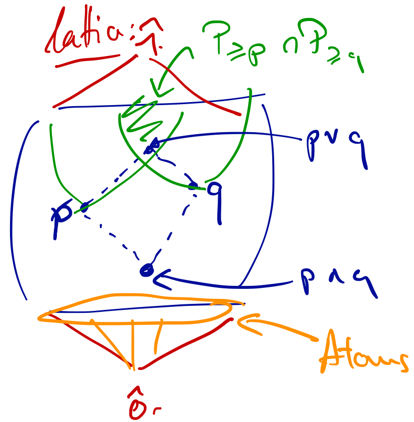
Program:

(1) Some poset theory

- $\mathcal{L}(\mathcal{A}) \longrightarrow$ matroid $g_{\mathcal{A}}$
- $\longleftarrow$
- Use $T_{g_{\mathcal{A}}}(x,y)$ to compute # of regions.

- **Poset theory** Let P be a partially ordered set

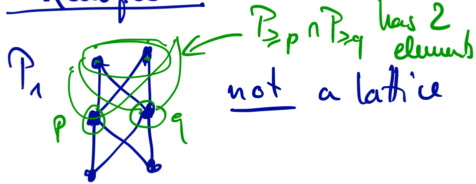
- P *lattice* if, for any $p, q \in P$
  - (a) $P_{\geq p} \cap P_{\geq q}$ has a unique min. element, which we then call *join* of p, q : "$p \vee q$"
  - (b) $P_{\leq p} \cap P_{\leq q}$ has a unique max. element called *meet* of p, q : "$p \wedge q$"



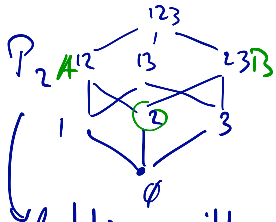
- Any finite lattice has a unique maximal element ($\hat{1}$) & a unique min. element ($\hat{0}$)
  i.e. "P bounded below"

- In any bounded-below poset P, the *atoms* are the $a \in P$ with $a > \hat{0}$. Set of atoms: $A(P)$

Examples.



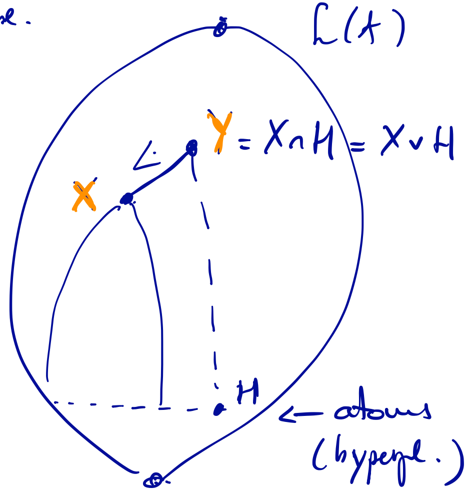
not a lattice



All subsets of $\{1,2,3\}$

lattice with $A \vee B = A \cup B$
$A \wedge B = A \cap B$

Example: If $\mathcal{A}$ arr. of hyps., $\underline{\mathcal{L}(\mathcal{A})}$ lattice.



Another interesting fact about $\mathcal{L}(\mathcal{A})$:

$$\boxed{X \lessdot Y \iff \exists H \in A(\mathcal{L}(\mathcal{A})),\ H \not\leq X, \text{ s.t. } Y = X \vee H}$$

Definition A finite lattice L is called **geometric** if it satisfies; $\forall x, y \in L$

$$(G) \quad \boxed{x \lessdot y \iff \exists p \in A(L),\ p \not\leq x, \text{ s.t. } y = x \vee p}$$

Example: Every $\mathcal{L}(\mathcal{A})$ is a geometric lattice.

Coming up: Geometric lattices $\xrightarrow{\text{together}}$ matroids. $\boxed{3}$

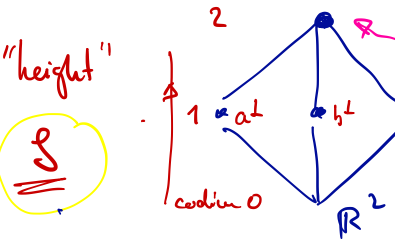
you?

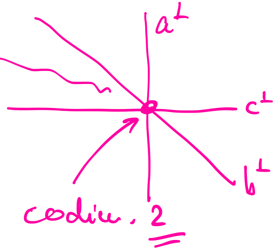
Intuition: $\mathcal{L}(\mathcal{A}) \leftarrow$ Arrangements of hyperplanes $\leftarrow$ normal vectors to the hyps. $\rightarrow$ matroid rank function

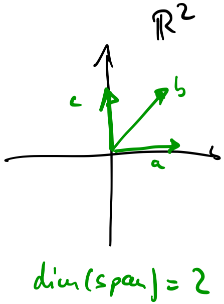
codimension of intersection of normal planes — dim. of span — rank

$\S$?

matroids can have $r(x)=0$ ⚠





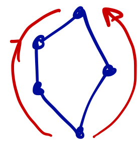


$r(a,b,c) = 2$

matroid on a, b, c

First task: Given any (abstract) geom. lattice, construct a (well-defined) "height function".

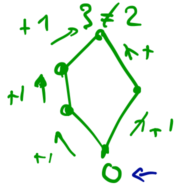
⚠ DO NOT WANT:



<u>Definition</u> A rank function for a poset P is any $\rho : P \to \mathbb{N}$ with

(i) $\rho(x) = 0$ if x minimal element of P

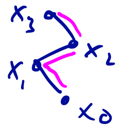
(ii) $\rho(x)+1 = \rho(y)$ if $x \lessdot y$ in P ⟵ Good! this forbids



<u>Note</u>: If P bounded below, ρ unique!

<u>Terminology</u>: A <u>chain</u> in P is any totally ordered $\omega = \{x_0 < x_1 \cdots < x_k\} \subseteq P$.

The <u>length</u> of ω is k:



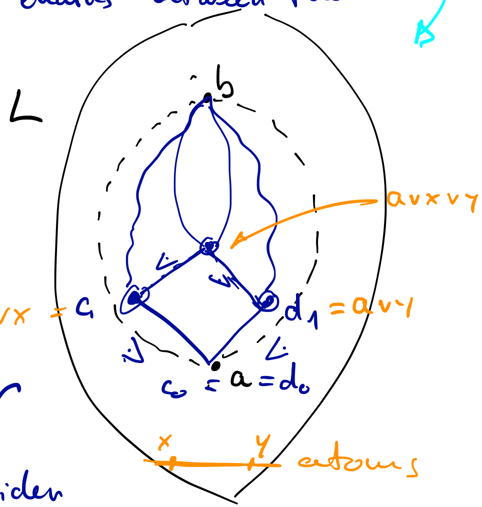
has length 3.

Geometric lattices have a $\hat{0}$, so there is only one candidate for a rank function: to set

$$\rho(y) = \text{length of } \underline{\text{any}} \text{ chain } \hat{0} \lessdot x_1 \lessdot \dots \lessdot x_k = y$$

→ we have to check that this is well-defined!

Lemma: In a geom. lattice any two maximal chains between the same elements have same length.

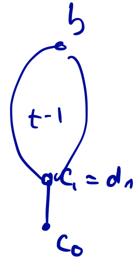


Proof Let L geom. l. Prove by induction on $t \geq 1$

$(*_t)$: For all $a,b \in L$, if one max. chain from a to b has length t, then all of them do.

$\underline{t=1}$: A max. chain $a \to b$ has length $1 \iff$ b–a $\; a \lessdot b$ ✓

$\underline{t \geq 2}$; suppose $(*_r)$ true for every $r < t$. Consider two maximal chains $a = c_0 \lessdot c_1 \cdots \lessdot c_t = b$, $a = d_0 \lessdot d_1 \lessdot \cdots \lessdot d_s = b$

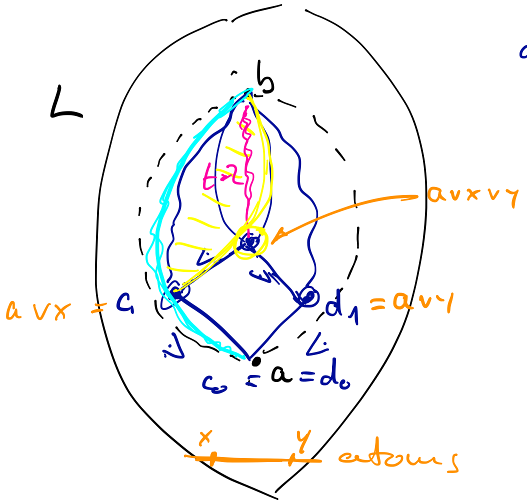


- **Case 1**: $c_1 = d_1$ – by $(*_{t-1})$ every chain $(c_1 = d_1) \to b$ has length $t-1$.

- **Case 2**: $c_1 \neq d_1$ By (G) there are $x, y \in A(L)$ with $c_1 = a \vee x$, $d_1 = b \vee y$

  Now: $x \nleq d_1$, since otherwise $c_1 = a \vee x \leq d_1$ ↯. Therefore, (G) implies

$$\boxed{c_1 \vee d_1 = a \vee x \vee y > d_1, c_1}$$



$a = c_0 < \cdots < c_t = b$

$a = d_0 < \cdots < d_s = b$

By $(*_{t-1})$ in $c_1 \to b$

In part: any maximal chain

$c_1 < a \vee x \vee y < \sim\sim\sim b$

has length $t-1$. So

$\sim\sim\sim$ has length $t-2$

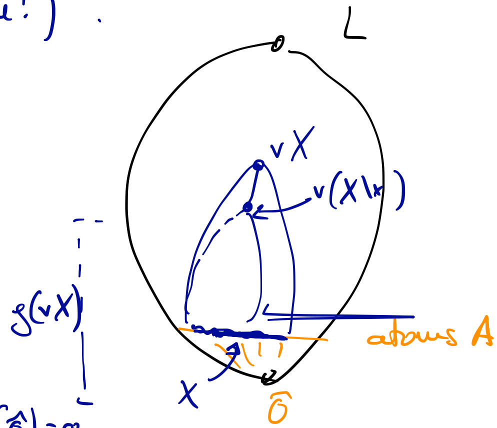
Analogously "on the r.h.s", $\sim\sim\sim$ has length $t-2$

$\Rightarrow$ both "c_i" and "d_i"-chains length t. $\boxtimes$

<u>Corollary – "Bottom line"</u>: In every geom. lattice L, the function

$$g: L \to \mathbb{N}, \qquad x \longmapsto g(x) := \text{length of } \underline{\text{any}} \text{ max. chain } \hat{0} \to x$$

is a rank function for L (& unique!).



For a set $X \subseteq A(L)$ look at $\vee X$

<u>Lemma</u> L, g, X as above. Then $g(\vee X) \le |X|$

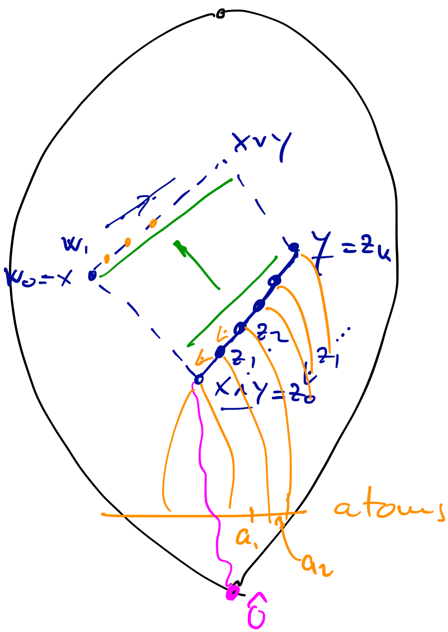
<u>Pf</u>: Induct on $|X|$. If $|X| = 0$, then $X = \emptyset$, and $\vee\emptyset = \hat{0}$, so $g(\vee X) = g(\hat{0}) = 0$.

If $|X| > 0$, pick $x \in X$ and notice: either $\vee(X \setminus x) = \vee X$ or $x \not\le \vee(X\setminus x)$, and then by (G) $\vee(X\setminus x) \lessdot \vee X$.

In either case: $$g(\vee X) \le \underbrace{g(\vee(X\setminus x))}_{\text{by i.Hyp.} \ \le |X\setminus x| = |X|-1} + 1 \qquad \blacksquare$$

Clearly if $Y \subseteq X \subseteq A(L)$, $\quad \vee Y \le \vee X$, so $\rho(\vee Y) \le \rho(\vee X)$.

We have to check $(r2)$



Lemma L geom. lattice, $x, y \in L$. Then

$$\rho(x) + \rho(y) \geq \rho(x \wedge y) + \rho(x \vee y)$$

Pf: $\underline{\rho(y) - \rho(x \wedge y)} = k$, for $x \wedge y = z_0 \lessdot z_1 \lessdot \cdots \lessdot z_k = y$

By (G) let $a_1 \dots a_k$ with $z_i = z_{i-1} \vee a_i$

Now "translate" the chain $(z_i)_i$:

$w_0 = x$, $\quad w_1 = x \vee a_1$, $\dots$ $w_i = w_{i-1} \vee a_i$ $\quad \dots \quad w_k$

⚠ not guaranteed: $a_i \not\leq w_{i-1}$, but any case, by (G): $\underline{w_i = w_{i-1} \text{ or } w_i \gtrdot w_{i-1}}$

Also: $w_k = \overbrace{x \vee a_1 \vee a_2 \cdots \vee a_k = x}^{x \geq x \wedge y} \vee \underbrace{(x \wedge y) \vee a_1 \vee \cdots a_k}_{= y} = x \vee y$

$\Rightarrow \rho(x \vee y) - \rho(x) =$ length of chain $w_0, w_1 \dots \leq k = \rho(y) - \rho(x \wedge y)$ $\quad \Rightarrow$ ∎

$\leq$

Idea: set up correspondence

$$\boxed{\text{matroids on ground set } E} \longleftrightarrow \boxed{\text{Geometric lattices of subsets of } E}$$

Proposition: Let E finite set, let $L \subseteq 2^E$, partially ordered by inclusion and such that $E \in L$. Suppose that L is a geometric lattice, ρ its rank fn. Then, for every $X \subseteq E$ there is a unique minimal $X' \in L$ with $X \subseteq X'$ and the extension r of ρ on 2^E given by $r(X) := \rho(X')$ is a matroid rank fn.

Example $E = \{1, \dots, 5\}$

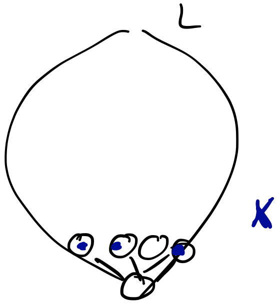
L:
$$\begin{tikzcd}
 & \{1,2,3,4,5\} & \\
\{1,2,3\} \arrow[ur, dash] & \{1,4\} \arrow[u, dash] & \{1,5\} \arrow[ul, dash] \\
 & \{1\} \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] &
\end{tikzcd}$$

ρ

$r: 2^{[5]} \to \mathbb{N}$

$r(\{1\}) = \rho(\{1\}) = 0$ $\quad$ 1 is a loop.

$r(\underbrace{\{4,5\}}_{X}) = r(\{4,5\}') = r([5]) = 2$



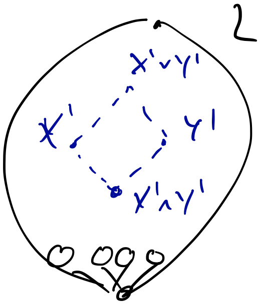
<u>Proposition</u>: Let E finite set, let $L \subseteq 2^E$, partially ordered by inclusion and such that $E \in L$. Suppose that L is a geometric lattice, ϱ its rank fn. Then, for every $X \subseteq E$ there is a unique minimal $X' \in L$ with $X \subseteq X'$ and the extension r of ϱ on 2^E given by $r(X) := \varrho(X')$ is a matroid rank fn.

<u>Proof</u>: (r1) trivial since $X \subseteq Y$ implies $X' \subseteq Y'$, so $r(X) = \varrho(X') \leq \varrho(Y') = r(Y)$.

(r0): $\varrho \geq 0$, hence $r(X) \geq 0$ $\forall X$. Moreover for any X consider all atoms $A_1 \dots A_k$ s.t. $A_i \cap X \neq \emptyset$ $\forall i$. Then $X' \leq \underline{A_1 \vee \dots \vee A_k}$, thus $r(X) = \varrho(X') \leq k \leq |X|$

(r2) — turn page.

Proposition: Let E finite set, let $L \subseteq 2^E$, partially ordered by inclusion and such that $E \in L$. Suppose that L is a geometric lattice, ρ its rank fn. Then, for every $X \subseteq E$ there is a unique minimal $X' \in L$ with $X \subseteq X'$ and the extension r of ρ on 2^E given by $r(X) := \rho(X')$ is a matroid rank fn.



contd $\rightarrow$ (r2) Take $X, Y \subseteq E$ consider X', Y'

Goal: estimate $\underbrace{r(X \cap Y)}_{\rho((X\cap Y)')}$, $\underbrace{r(X \cup Y)}_{\rho((X\cup Y)')}$ via $\rho(X' \wedge Y')$, $\rho(X' \vee Y')$

Note: • $\boxed{X' \wedge Y' \geq \underbrace{(X \cap Y)'}_{\subseteq X',\ \subseteq Y'}}$ (1) –

• $X' \vee Y'$ minimal in L containing X', Y'
$(X \cup Y)'$ minimal in L containing X, Y $\quad (\cup\!\!\mid\ \ \cup\!\!\mid)$

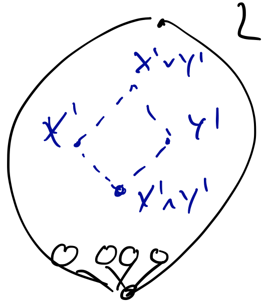
$\Big\}$ $X' \vee Y' \geq (X \cup Y)'$

Moreover: $X' \vee Y' \leq (X \cup Y)'$ because $X \subseteq X \cup Y \subseteq (X \cup Y)'$, but X' minimum in L with $X \subseteq X'$, thus $X' \subseteq (X \cup Y)'$ $\circledast$

$(\cup\!\!\mid$ X', $\cup\!\!\mid$ $Y')$

$\Longrightarrow \boxed{X' \vee Y' = (X \cup Y)'}$ (2) –

Proposition: Let E finite set, let $L \subseteq 2^E$, partially ordered by inclusion and such that $E \in L$. Suppose that L is a geometric lattice, ϱ its rank fn. Then, for every $X \subseteq E$ there is a unique minimal $X' \in L$ with $X \subseteq X'$ and the extension r of ϱ on 2^E given by $r(X) := \varrho(X')$ is a matroid rank fn.



Note: $\boxed{X' \wedge Y' \geq \underbrace{(X \cap Y)'}_{\leq X',\ \leq Y'}}$ ① $\implies$ $\boxed{X' \vee Y' = (X \cup Y)'}$ ②

$$r(X) + r(Y) \overset{\text{def}}{=} \varrho(X') + \varrho(Y') \overset{\text{lemma}}{\geq} \varrho(\underbrace{X' \wedge Y'}_{\geq (X \cap Y)'}) + \varrho(\underbrace{X' \vee Y'}_{=(X \cup Y)'})$$

$$\overset{①②}{\geq} \varrho((X \cap Y)') + \varrho((X \cup Y)')$$

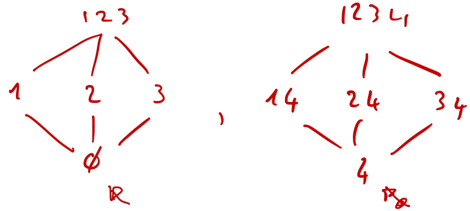
$$\overset{\text{def}}{=} r(X \cap Y) + r(X \cup Y)$$

$\blacksquare$

... This way: **Geom. lattices of subsets (of E)** $\rightleftarrows$ **Matroids on E**

Notice: $\downarrow$ ← surj, not injective

"Geometric lattices"



Now: ←

Definition: Let E finite set, $r: 2^E \to \mathbb{N}$ matroid rank.

Define "closure operator"

$$cl: 2^E \longrightarrow 2^E, \quad X \longmapsto \{e \in E \mid r(X \cup \{e\}) = r(X)\}$$

(Idea: $cl(X) \approx X'$)

Call $X \subseteq E$ closed if $X = cl(X)$, call $\mathcal{L}_r$ the poset of all closed sets, ordered by inclusion.

⚠ current version of script uses ϱ for r - I'll update.

Definition: Let E finite set, $r: 2^E \to \mathbb{N}$ matroid rank.

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Call $X \subseteq E$ closed if $X = cl(X)$, call $\mathcal{L}_r$ the poset of all closed sets, ordered by inclusion

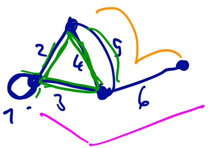
Goal ~~Idea~~: prove $\mathcal{L}_r$ is GL.

Homework: ① read § 3.3, & report corrections, ask questions

② Do exercises III 1 – III 3

Wait until evening ($g \to r$)

Example (#3.3.2 in Lecture Notes)



matroid on $[6]$,

rank = size of maximal acyclic set of edges,

forest: $r(1) = 0$
$r(45) = 1$

$cl(\{1\}) = \{1\}$

$cl(\{6\}) = \{6, 1\}$

⚠ 1 is in every closure!

$cl(\{4\}) = \{1, 4, 5\}$

$r(1,4,5) = 1$

$cl(\{2,3\}) = \{1,2,3,4,5\}$

$\mathcal{L}_r$: