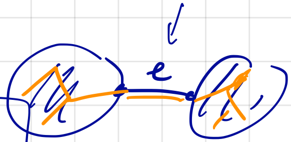
COMBINATORIAL POLYNOMIALS – 2.04.2020

First: III.2 III.4 & II.5

III.2 Given: $r: 2^E \to \mathbb{N}$, $e \in E$ is isthmus.



$r(e)=1$; $\boxed{r(A\cup e) = r(A)+1 ,\ \forall A \subseteq E\setminus e}$

def. of "e is isthmus"

Prove: $\mathcal{L}_{/e} \simeq \mathcal{L}_{\setminus e}$ sets of closed sets, ordered by incl.

$A \subseteq E$ is closed if $A = cl(A) = \{x \in E \mid r(A\cup x) = r(A)\}$

Now compare $r_{/e}$, $r_{\setminus e}$. Both on ground set $E\setminus\{e\}$.

For $A \subseteq E\setminus\{e\}$,

$$r_{\setminus e}(A) \overset{def}{=} r(A)$$

$$r_{/e}(A) \overset{def}{=} r(A\cup\{e\}) - r(\{e\})$$

$$= r(A) + 1 - 1$$

$$= r(A)$$

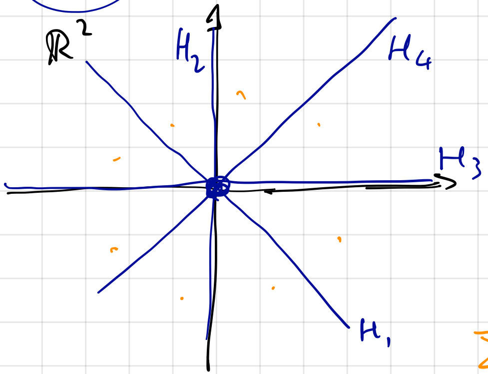
e isthmus

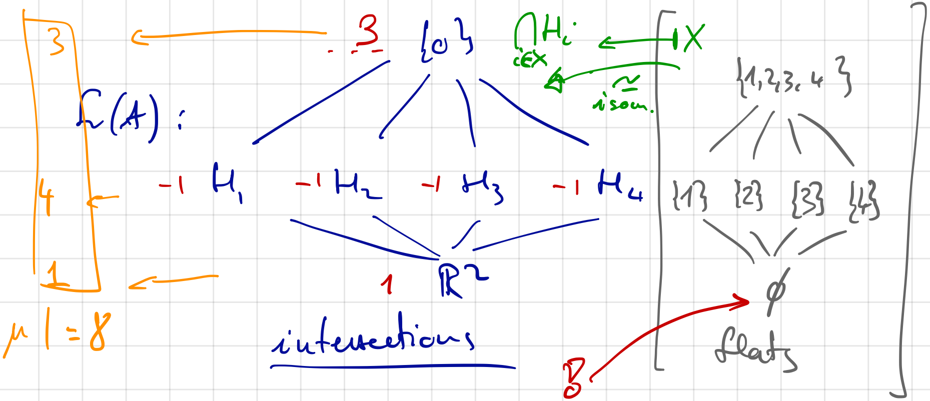
In general, (finite) posets P, Q are isomorphic if $\exists\, f: P \to Q$ order-preserving bijection

Here $f: \mathcal{L}_{\mathbb{R}} \to \mathcal{L}_{\mathbb{R}}$ is identical map, orderpreserving:

$A \longmapsto A$ for $A_1 \subseteq A_2$ we have $\underbrace{f(A_1)}_{= A_1} \subseteq \underbrace{f(A_2)}_{= A_2}$

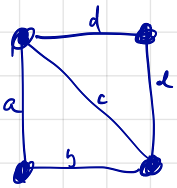
III.4 I meant: $H_1: x+y=0$, $H_2: x=0$, $H_3: y=0$, $H_4: x-y=0$





Look at III.5

G:

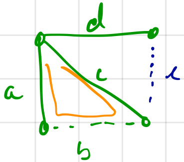


$E = \{a, b, c, d, e\}$

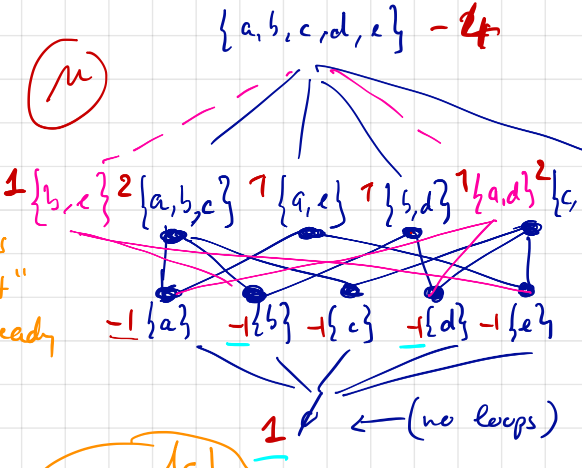
Flats of r_G :

taking closure $\approx$ adding all edges that "close a circuit" with what is already in ~~the~~ set.

$cl(\{a, c, d\}) \ni b$



μ



← (no loops)

$\Rightarrow$ Δ_G has 18 regions

-2

8

-5

1

$\downarrow$

coeff. of χ_G

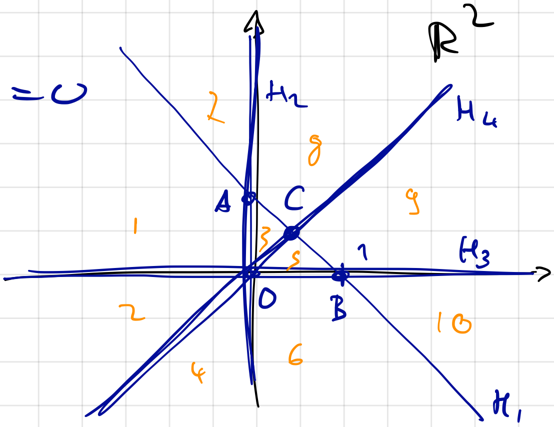
Quick look at III.6: "Möbius function" of L, $\mu : L \to \mathbb{Z}$

$$\mu(\hat{0}) = 1 \;, \quad \sum_{y \leq x} \mu(y) = 0 \quad \forall x \in L$$

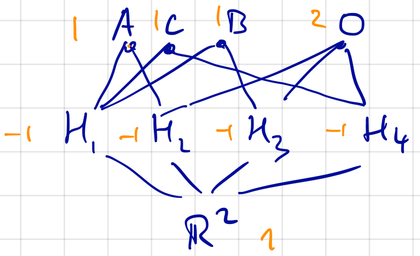
Example

Curiosity: The "wrong" III.4:

· $x+y=1$, $x=0$, $y=0$, $x-y=0$



intersections:



μ : $|\mu|$

5

4

1

$\sum |\mu| = 10$

NOT A LATTICE!

Ch. 4 – Signed graphs!

Review: $G = K_n$ complete graph

$\mathcal{A}_G$ arrangement in $\mathbb{R}^n$, with normals $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ +1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = e_i - e_j$

std. basis of $\mathbb{R}^n$

a set of positive roots for A_n

$\chi_G(t) = t(t-1)\cdots$

$|\mathcal{R}(\mathcal{A}_G)| = |\chi_G(-1)| = n!$

= cardinality of Coxeter group of type A_{n-1}

($\simeq$ perm-group on n elts)

Q: can we treat other Coxeter types?

A: YES, via Zaslavsky's theory of signed graphs

Definition A loopless *signed graph* is $\Sigma = (G, \sigma)$

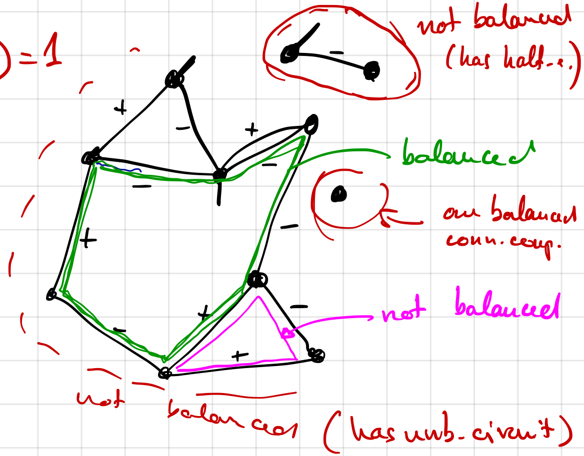
where $G = (V, E, h: E \to V, t: E_r \to V)$, $\sigma: E_r \to \{+,-\}$

$$E = E_r \uplus E_{1/2}$$

regular edges ↗ ↖ half-edges

$b(E) = 1$

$b(\text{green edges}) =$



"loopless" means $|\{h(e), t(e)\}| = 2 \quad \forall e \in E_r$

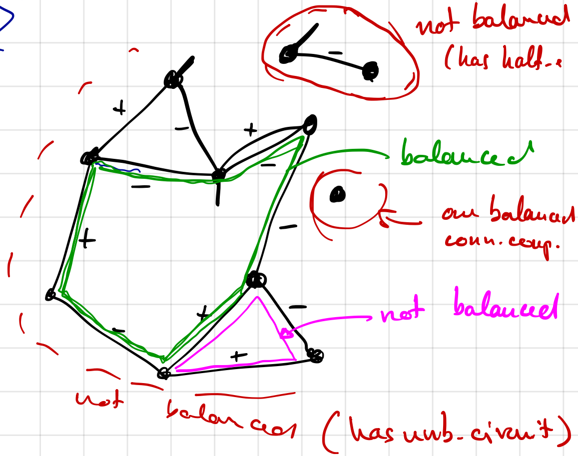
Balance: Every path $p: e_1, e_2, e_3 \dots$

has a sign $\sigma(p) = \sigma(e_1)\sigma(e_2)\dots$

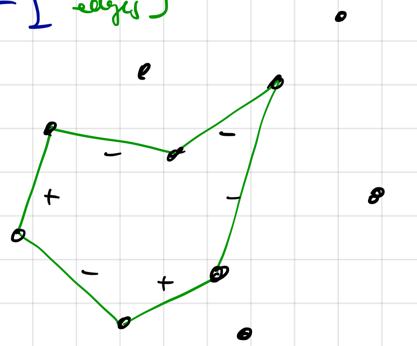
A circuit C of Σ is balanced if $\sigma(C) = +$

A connected component of Σ is balanced if has no half-edges & all circuits are balanced. For $A \subseteq E$, call $b(A) = \#$ of balanced connected components on $\Sigma[A] := (G[A], \sigma|_A)$.

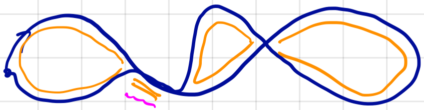
Σ



$\Sigma[\text{green edges}]$

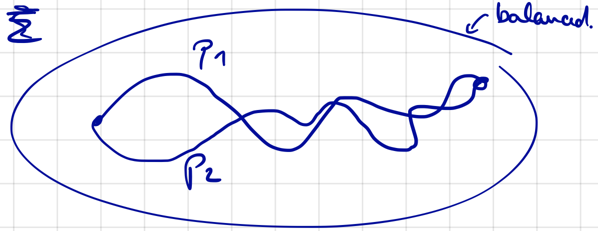


$$b(\text{green}) = 6$$



Remark: Any cycle in a balanced component has positive sign! Pf: cycle is disj union of circuits & "doubly traversed edges", so the sign of the cycle is the product of the $\underbrace{\text{signs of circuits}}_{+}$ & $\underbrace{\text{doubly traversed edges}}_{(-1)^2=+1}$

Lemma: Let p_1, p_2 two paths with same endpoints, in a balanced comp. of a signed graph.



Then $\sigma(p_1) = \sigma(p_2)$, i.e., the product of the signs of edges of p_1 equals the product of signs of edges of p_2.

Pf: $p_1 \cup p_2$ is cycle, and by remark above $\sigma(p_1 \cup p_2) = +$

$$+ = \sigma(p_1 \cup p_2) = \sigma(p_1)\,\sigma(p_2) \Rightarrow \sigma(p_1) = \sigma(p_2)^{-1} = \sigma(p_2) \qquad \blacksquare$$

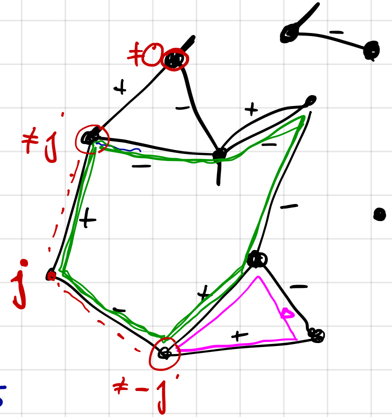
Colorings of signed graphs

Let $k \in \mathbb{N}$ and $[[k]] := \{-k, -(k-1), \dots, 0, 1, \dots, k\}$.

A (signed) k-coloring of a loopless signed graph Σ is an assignment $\gamma : V \longrightarrow [[k]]$ such that

(i) $\gamma(h(e)) \neq \sigma(e)\, \gamma(t(e))$ for all $e \in E_r$

(ii) $\gamma(h(e)) \neq 0$ for all $e \in E_{1/2}$

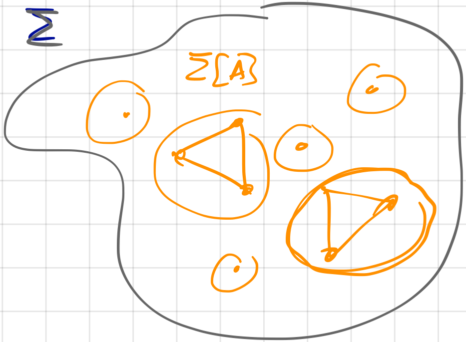


As in unsigned case:

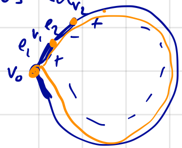
- $\chi_\Sigma^{\mathbb{N}}(2k+1) = \#$ of signed k-colorings of Σ
- call "precoloring" <u>any</u> $\gamma : V \longrightarrow [[k]]$.
- For $A \subseteq E$ set
$$\mathrm{Pre}_\Sigma(A) = \left\{ \gamma : V \to [[k]] \;\middle|\; \begin{array}{ll} \gamma(h(e)) = \sigma(e)\,\gamma(t(e)) & e \in A \cap E_r \\ \gamma(h(e)) = 0 & e \in A \cap E_{1/2} \end{array} \right\}$$

- For $A \subseteq E$ set
$$\mathrm{Pre}_\Sigma(A) = \left\{ \chi : V \to [[k]] \;\middle|\; \begin{array}{ll} \chi(h(e)) = \sigma(e)\,\chi(t(e)) & e \in A \cap E_r \\ \chi(h(e)) = 0 & e \in A \cap E_{1/2} \end{array} \right\}$$

Take $A \subseteq E$. Any $\chi \in \mathrm{Pre}_\Sigma(A)$ will have value 0 on every unbalanced component of $\Sigma[A]$. In fact



Case 1: the unbal. component contains an unbalanced cycle C: $e_1 e_2 \dots$



$$\chi(v_0) = \sigma(e_1)\,\chi(v_1) = \dots$$
$$= \dots = \sigma(e_1)\,\sigma(e_2)\cdots\sigma(e_n)\,\chi(v_n)$$
$$= \sigma(C)\,\chi(v_0) \quad \Rightarrow \quad \chi(v_0) = 0 \quad \Rightarrow \quad \chi \equiv 0 \text{ on whole component.}$$

Case 2: the unbal. component contains a half-edge. By the second condition χ is 0 on one vertex – hence on all vertices of the component.

In summary:
$$\left|\mathrm{Pre}_\Sigma(A)\right| = (2k+1)^{b(A)}$$

no choice for unbalanced comps. One "degree of freedom" for every balanced comp.

Inclusion - exclusion as in the "classical" case (Chapter 1!)

given:

$$\chi_{\Sigma}^{\mathbb{N}}(2k+1) = \underbrace{(2k+1)^{|V|}}_{\text{all precolorings}} + \sum_{\emptyset \neq A \subseteq E} (-1)^{|A|} \underbrace{|\mathrm{Pre}_{\Sigma}(A)|}_{(2k+1)^{b(A)}}$$

$b(\emptyset)$

$$= \sum_{A \subseteq E} (-1)^{|A|} (2k+1)^{b(A)}$$

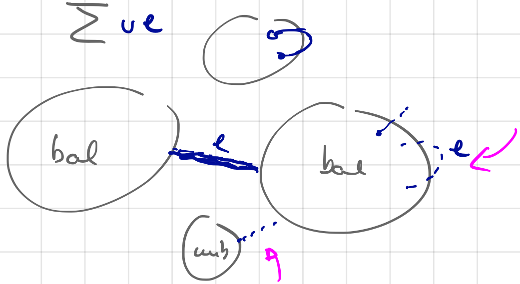
$\Rightarrow$ $\chi_{\Sigma}^{\mathbb{N}}$ agrees with (the evaluation at odd numbers of) a polynomial

$$\chi_{\Sigma}(t) = \sum_{A \subseteq E} (-1)^{|A|} t^{b(A)}$$

Hopefully "matroidal"

Balance and matroids

$\Sigma \cup e$



Definition: Let Σ be a loopless signed graph on n vertices, with set of edges E. Define

$$r_\Sigma : 2^E \longrightarrow \mathbb{N}, \quad A \longmapsto r_\Sigma(A) := n - b(A)$$

$\boxed{\text{G.O.}}$ When adding an edge e: either a balanced comp. becomes unbalanced (if e half-edge, or e closes an unbalanced cycle, or e joins a balanced with an unbalanced component), or two balanced components get joined in a bigger balanced comp.

In any case: $\quad b(A \cup e) = b(A)$ or $b(A) - 1$, $\quad$ hence:

$$r_\Sigma(A) \leq r_\Sigma(A \cup e) \leq r_\Sigma(A) + 1 .$$

Theorem: For every loopless signed graph Σ, the function r_Σ is the rank function of a matroid on E. ✓

Proof check the axioms.

(r0) $0 \leq r_\Sigma(A) \leq |A|$ for all $A \subseteq E$

clear: $b(A) \leq n = |V|$

Let Σ_0: graph on V without edges.

then $r_\Sigma(\emptyset) = n - n = 0 = r(\Sigma_0)$

Now add all edges in A to Σ_0, one by one: By $\boxed{\text{G.O.}}$, at every step increase ≤ 1, thus $r_\Sigma(A) \leq |A|$.

(r1) Enough to prove: $r_\Sigma(A) \leq r_\Sigma(A \cup \{e\})$, this is directly from $\boxed{\text{G.O.}}$

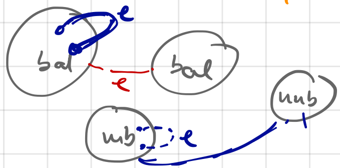
(r2) $r_\Sigma(A)+r_\Sigma(B) \geq r_\Sigma(A\cap B)+r_\Sigma(A\cup B)$ for $A,B\subseteq E$.

Note: it is enough to prove the inequality for $A=(A\cap B)\cup\{e\}$, $B=(A\cap B)\cup\{f\}$ (Exercise)

So, we prove: $r_\Sigma(A\cup\{e\})+r_\Sigma(A\cup\{f\}) \geq r_\Sigma(A)+r_\Sigma(A\cup\{e,f\})$,

or equivalently: $\underbrace{r_\Sigma(A\cup\{e\})-r_\Sigma(A)}_{(*)} \geq \underbrace{r_\Sigma(A\cup\{e,f\})-r_\Sigma(A\cup\{f\})}_{(\Delta)\ \leq 1 \text{ by G.O.}}$ $\forall A,e,f$

Observe: If $(*)=1$, the inequality is trivial!

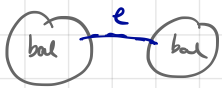


So suppose $(*)=0$ – i.e. e either has two ends, both in a bal. component, or e has only ends in unbalanced comp.

B.w.o.e. assume $(\Delta)=1$, i.e.: adding e to $\Sigma[A\cup f]$ decreases # of bal. com.

(1) e connects two bal. comp. of $\Sigma[A\cup f]$ – contradiction to: e does not connect two different bal. comp. of $\Sigma[A]$

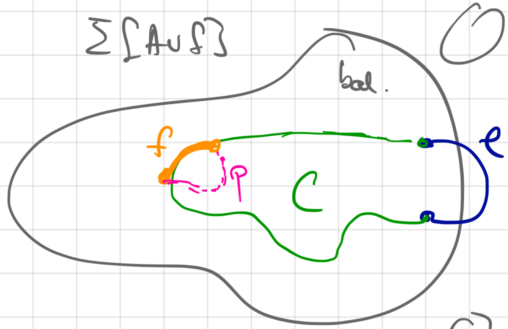
$\Sigma[A\cup f]$



(2) e unbalances a comp. of $\Sigma[A\cup\{f\}]$

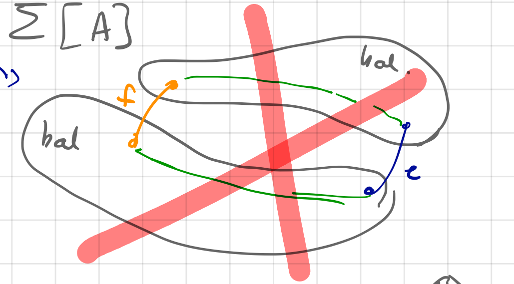
(2.1) e is a half-edge in a bal. comp of $\Sigma[A\cup\{f\}]$
$\Rightarrow$ e is " " " " " " $\Sigma[A]$. as well ↯

So: (2.2) e "closes" an unbalanced circuit C in a bal. comp. of $\Sigma[A \cup f]$



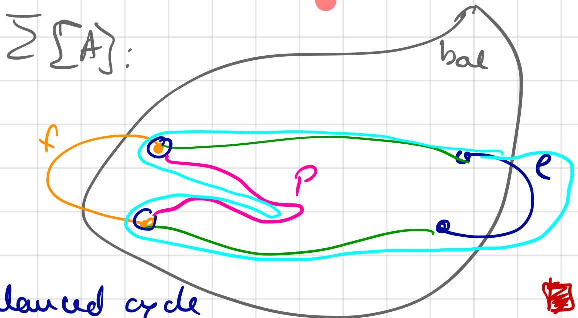
Note:

- $C \ni f$! Otherwise C already contained in a bal comp. of $\Sigma[A]$.
- If f joins two bal. comp. of $\Sigma[A]$, then e does, too, hence $r_\Sigma(A \cup e) > r_\Sigma(A)$ ↯



Then: both ends of f in same balanced comp. of $\Sigma[A]$.

Thus, these ends are connected by a path p with $\sigma(p) = \sigma(f)$ (by Lemma!)



BUT THEN: $(C \setminus f) \cup p$ is an unbalanced cycle in a bal. comp. of $\Sigma[A \cup e]$. ↯

sign: $\sigma(p)\sigma(\sim)\sigma(e) = \sigma(f)\sigma(\sim)\sigma(e) = \sigma(C)$

Take a deep breath:

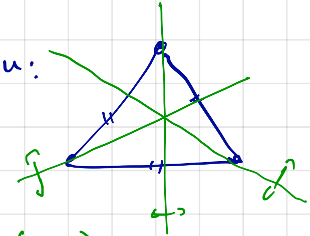
Σ - loopless signed graph, n vertices, E edges.

$r_\Sigma : 2^E \to \mathbb{N}$, $\quad r_\Sigma(A) = n - b(A)$ defines a matroid.

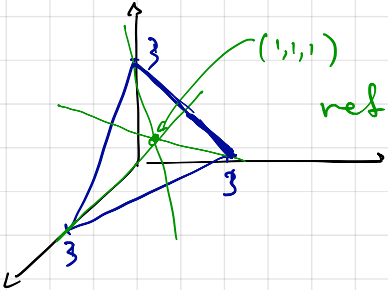
<u>Idea</u>: find a "representation" of r_Σ by a hyp. arrangement, so that $T_{r_\Sigma}(1-t, 0)$ has an interpretation (& hence $\chi_\Sigma(t)$)

Quick peek into (finite) Coxeter groups.

regular polygon:



green reflections generate group of symmetries of the equilateral triangle.



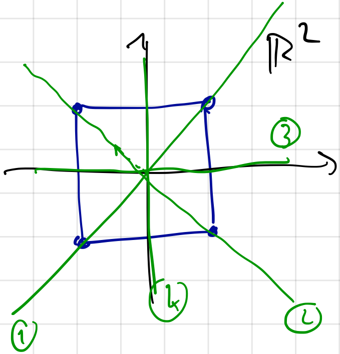
reflection planes: normals $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$\Rightarrow \{e_i - e_j\}_{i \neq j}$ set of normals for the "reflection planes" of (d-dim.) simplex (equil. triangle, tetrahedron, ...)

conv $\{e_i\}_i$ in $\mathbb{R}^{d+1}$

$\hookrightarrow$ group of orthogonal symmetries of d-dim. tetrahedron is "A_{d+1}"

Other regular polygon:



Reflection planes:

① : $(e_1 - e_2)^\perp$

② : $(e_1 + e_2)^\perp$

③, ④ $e_1^\perp$ $e_2^\perp$

Coxeter type B_n: reflections @ planes: $\begin{cases} (e_i - e_j)^\perp, \ (e_i + e_j)^\perp & 1 \le i < j \le n \\ e_i^\perp & 1 \le i \le n \end{cases}$

regular graphs!

$e_i - e_j$

$e_i + e_j$

e_i

A_n: $(e_i - e_j)^\perp$

C_n: $(e_i - e_j)^\perp$ $(e_i + e_j)^\perp$, $(2e_i)^\perp$

D_n: $(e_i - e_j)^\perp$ $(e_i + e_j)^\perp$

& I_n (dim 2), G_6, E_7, E_8, H_7 (finitely many)