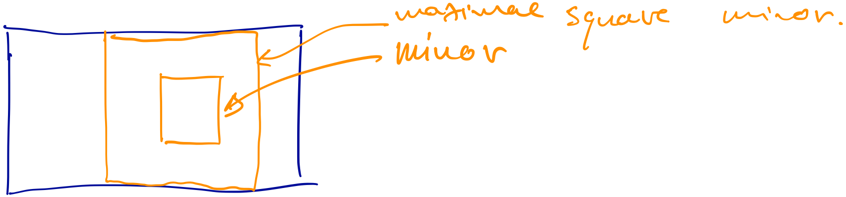
Matrix $A =$ 

A totally unimodular if every square minor has det. $0, 1, -1$

A unimodular if every maximal square minor has det. $\cancel{0,} \pm 1$.

Main example: Incidence matrices of graphs are totally unimodular

Remark: An integer, invertible square matrix B is unimodular iff B^{-1} is integral.

Pf. • If B unimodular, then $B^{-1} = \frac{1}{\det(B)} \underbrace{\mathrm{Adj}(B)}_{\text{integral.}}$

• If B^{-1} integral: $\det(B) = \frac{1}{\det(B^{-1})}$

Proposition: Let A integral matrix of full row rank: $A = {}_n\boxed{}_m$

A unimodular $\Leftrightarrow$ for every vector $b \in \mathbb{Z}^m$, $P_b := \{x \mid Ax = b,\ x \geq 0\}$ has only integral vertices

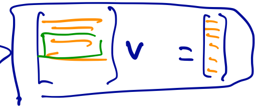
Proof: "$\Rightarrow$"

$$P_b = \left\{ \begin{bmatrix} -\mathbb{1} \\ \hline A \\ \hline -A \end{bmatrix} x \leq \begin{bmatrix} 0 \\ \hline b \\ \hline -b \end{bmatrix} \right\}$$

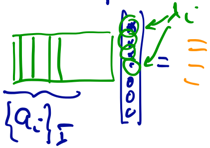
If v a vertex of P_b:

n lin. indep. constraints satisfied with equality @ v

$$v = \left\{ x \in P_b \;\middle|\; \equiv x = \equiv \right\}$$

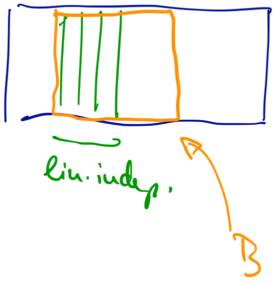
$\Rightarrow$ 

Claim: columns of A corresponding to nonzero comp. of v are lin. indep.



Otherwise there is lin. dep $\sum_{i=1}^{k} \lambda_i a_i = 0$ $\hookrightarrow$ not all $= 0$

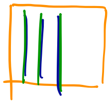
and $\underbrace{v + \varepsilon \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\text{in the same face as } v}$ has $A\left(v + \varepsilon \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \\ 0 \end{bmatrix}\right) = Av + 0$ $\lightning$

So: A 

$v: \begin{bmatrix}\neq 0\\ 0\\ \vdots\\ 0\end{bmatrix}$

Complete $\{a_i\}_I$ to a column-basis of A to a square, integer maximal minor B

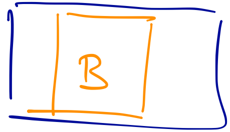
With: $\boxed{B^{-1} b = v}$ at every nonzero comp. of v.

Note $Bv =$  $\begin{bmatrix}\neq 0\\ 0\\ \vdots\\ 0\end{bmatrix} =$ "b" (at some components, i.e., those where $v \neq 0$)

Now we know: B unimodular $\Rightarrow$ B^{-1} integral! (remark),

therefore $B^{-1}b$ integral, and so: nonzero comp. of v are integral ∎

$\Leftarrow$ "P_b integral $\forall b \Rightarrow A$ unimodular"   Let B max. minor of A



Pick a nonsingular max. minor B

Prove: B^{-1} integral (by Remark this is enough)

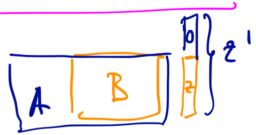
Enough to prove: $B^{-1}c \in \mathbb{Z}^n$ for all $c \in \mathbb{Z}^m$.   (think $c = e_i$ standard basis vector)

ith column of B^{-1}

$$B^{-1} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} \leftarrow e_i = \begin{bmatrix} \\ \\ \end{bmatrix}$$

Let $c \in \mathbb{Z}^n$, choose $y \in \mathbb{Z}^n$ s.t. $z := y + B^{-1}c \geq 0$.

Then $Bz = By + c \in \mathbb{Z}^n$, and after "extending z by zero components":



$Az' = Bz = b$, and z' is vertex of P_b (satisfies n lin. indep. constraints with equality)

Thus: z' is integral, and so are z and $B^{-1}c$ ∎

Proposition: Let A integral matrix of full row rank: $A = n\,\underbrace{\boxed{}}_{m}$

A unimodular $\iff$ for every vector $b \in \mathbb{Z}^m$, $\boxed{P_b := \{x \mid Ax = b,\ x \geq 0\}}$ has only integral vertices

Moral proof:
- B inv, integral: unimodular $\iff$ B^{-1} integral
- Invertibility of maximal minors of A $\longleftrightarrow$ non-perturbability of vertices.

Theorem: For an integral matrix A TFAE

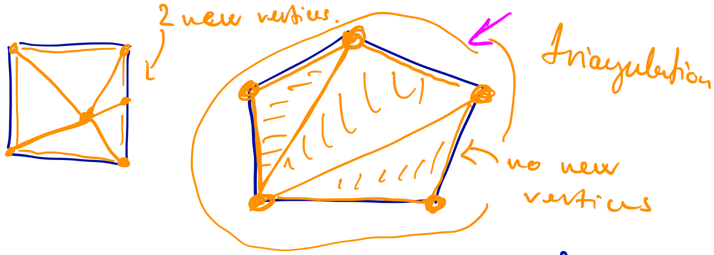
(i) A totally unimodular

(ii) All vertices of $\{x \mid Ax \leq b, x \geq 0\}$ are integral, for all integral b.

(iii) All vertices of $\{x \mid a \leq Ax \leq b, c \leq x \leq d\}$ are integral, for all integral a, b, c, d.

Proof: More matrix yoga. $\longrightarrow$ lecture notes.

TRIANGULATIONS



Recall $X \subseteq \mathbb{R}^d$ affinely indep. if $\dim(\mathrm{conv}(X)) = |X|-1$. **Simplex**: any conv. hull of an affinely indep. set of points.

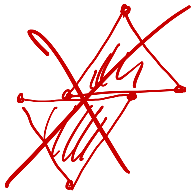
We want to properly define a "triangulation" as a subdivision of a polytope into simplices.

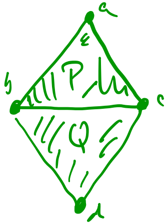
Definition: A polyhedral/polytopal **complex** is a collection K of polyhedra/polytopes s.t. (1) every face of every element of K is in K

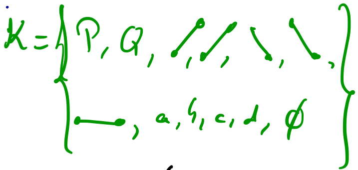
(2) For any two $P, Q \in K$, $P \cap Q$ is a face of both.

The support of K is

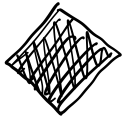
$$|K| = \bigcup_{P \in K} P$$





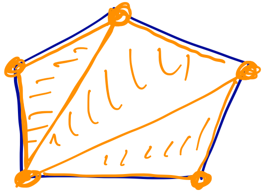


$|K|$

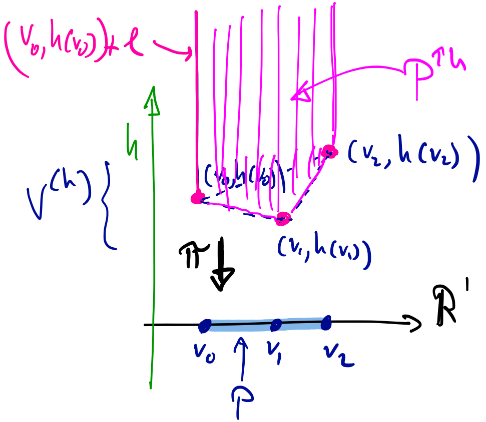


Definition Let P be a convex polytope. A **subdivision** of P is any polytopal complex K with $|K| = P$. A **triangulation** of P is any subdivision T of P s.t. every member of T is a simplex.

Goal today: every polytope has a triangulation
"with no new vertices"



Setup Let $V \overset{\text{finite}}{\subseteq} \mathbb{R}^d$, let $P = \mathrm{conv}(V)$.
wlog $\dim(P) = d$.



Let $h: V \to \mathbb{R}_{>0}$, "lift" V to a set

$$V^{(h)} := \{ (v, h(v)) \mid v \in V \} \subseteq \mathbb{R}^{d+1}$$

Moreover, let $\ell := \{ (0,t) \mid t \geq 0 \}$, and consider

$$P^{\uparrow h} := \underbrace{\mathrm{conv}\, V^{(h)}}_{\text{polyhedron}} + \underbrace{\ell}_{\text{cone}} \quad - \text{"epigraph" of } P.$$

Natural projection $\pi: \mathbb{R}^{d+1} \to \mathbb{R}^d$, $(x,t) \longmapsto x$. Note: $\pi(\ell) = 0$,
so $\pi(P^{\uparrow h}) = P$.

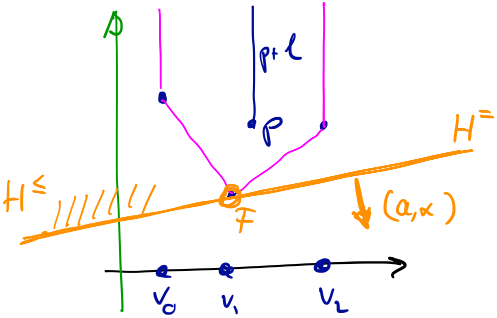
We know (2 weeks ago): $P^{\uparrow h}$ is a polyhedron.

Let us study supporting hyperplanes to $P^{\uparrow h}$.

Supporting hyperplane: $H^{=} = \left\{ (x,t) \in \mathbb{R}^{d+1} \;\middle|\; \langle (a,\alpha),(x,t)\rangle = b \right\}$

$H^{\leq} = \left\{ \ldots \ldots \langle (a,\alpha),(x,t)\rangle \leq b \right\}$

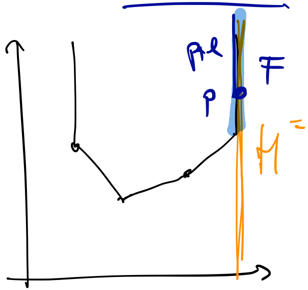
$\langle a,x\rangle + \alpha t$



<u>Lemma</u>: $P^{\uparrow h} \subseteq H^{\leq}$ implies $\alpha \leq 0$. If $H^{=}$ is supporting hyperplane, then face $F = P^{\uparrow h} \cap H^{=}$ bounded if and only if $\alpha < 0$.

<u>Pf</u>: If $P^{\uparrow h} \subseteq H^{\leq}$, then for every $p \in P^{\uparrow h}$ the whole $p+\ell \subseteq H^{\leq}$, i.e.

$$\underbrace{\langle a,p\rangle}_{\text{fix}} + \alpha t \leq b \quad \forall\, t \geq 0 \quad \Rightarrow \quad \alpha \leq 0$$

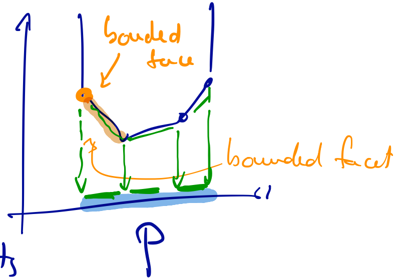


Now let $H^{=}$ be a supporting hyp,

$F = P^{\uparrow h} \cap H^{=}$ unbounded $\Leftrightarrow$ $F + \ell \subseteq F$, i.e. $\forall p \in F$: $p + \ell \subseteq F \subseteq H^{=}$. This means:

$$\langle a,p\rangle + \alpha \cdot t = b \quad \forall t \in \mathbb{R}_{\geq 0} \quad \Rightarrow \quad \alpha = 0.$$

Corollary Every bounded face of P^{1h} is a face of some bounded facet.



Pf: Recall (last week) that every proper face F is intersection of facets: $F = \bigcap_{i=1}^{k} F_i$ ← facets

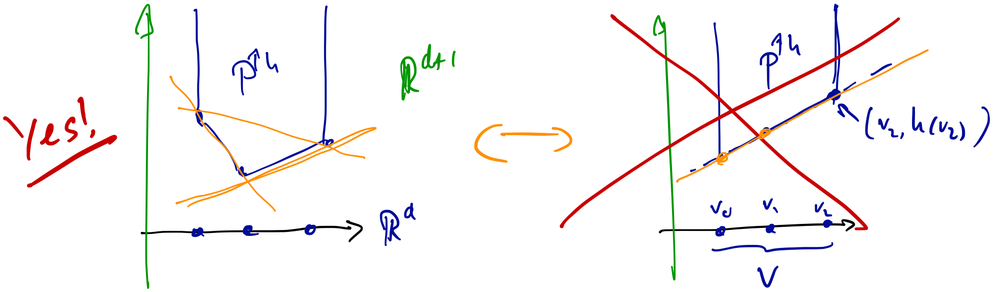
If all F_i unbounded, then for every $p \in F$

we have: $p \in F_i$, hence $p + \ell \subseteq F_i \quad \forall i$

Thus, $p + \ell \subseteq F$, so F is unbounded.

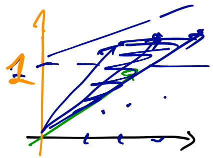
Claim follows by contraposition.

After the break: Tweak things s.t. $\boxed{\pi\left(\begin{matrix}\text{bounded}\\ \text{faces of } P^{1h}\end{matrix}\right)}$ triang. of P



Lemma: h can be chosen so that: if $\{v_0, \dots, v_d\} \subseteq V$ max. affinely indep. then the $(v_i, h(v_i))$ are affinely independent & they span a hyperplane that does not contain any other point of $V(h)$

Pf: Let $V = \{v_0, \dots, v_m\}$, consider any aff. indep. $I = \{i_0 \cdots i_d\} \in \binom{[m]}{d+1}$



$\{(v_i, h(v_i))\}_{i \in I}$ Aff. indep "automatically"

For — :

$$\underbrace{\begin{vmatrix} v_{i_0} & \cdots & v_{i_d} & v_j \\ h(v_{i_0}) & & h(v_{i_d}) & h(v_j) \\ 1 & & 1 & 1 \end{vmatrix}}_{p(I \cup \{j\})} \neq 0$$

Now the desired values of $h : V \to \mathbb{R}_{>0}$ are those for which

$$\underbrace{\prod_{I,j} \gamma(I \cup \{j\}) \neq 0}_{\text{(non-constant) polynomial in } h_i}$$

By analytic reasons, such an h exists. $\square$

Theorem: Every polyhedron $P = \mathrm{conv}(V)$ has a triangulation all of whose vertices are in V.

Proof: Choose h as in the Lemma. Consider

$$\mathcal{T} := \{ \pi(F) \mid F \text{ bounded face of } P^h \}$$

Obs: The vertices (0-dim. members) of $\mathcal{T}$ are (a subset of) V.

Let F be a bounded facet of P^h

choice of h (a) F is a simplex (by genericity of h)

F is bounded

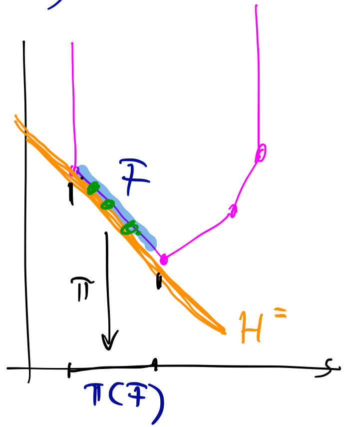
(b) The restriction of π to F is bijective (inverse is "explicit equation of $H^=$")

(c) For $X \subseteq F$, $\mathrm{conv}(\pi(X)) = \pi(\mathrm{conv}(X))$

(d) $\pi(F)$ is a simplex with vertices $\pi(\text{vert. of } F)$,

(e) Faces of $\pi(F)$ are exactly projections of faces of F.

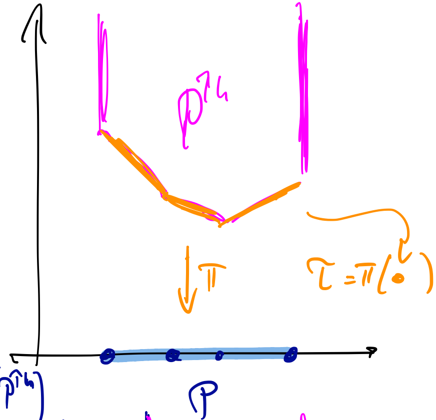
Previous corollary (f) $\mathcal{T} = \{ \text{faces of } \pi(F),\ F \text{ a bounded facet} \}$



Claim: $\tilde{\tau}$ is triangulation of P

(1) Every member of $\tilde{\tau}$ is a simplex.

-> clear by previous discussion (items (d),(f))



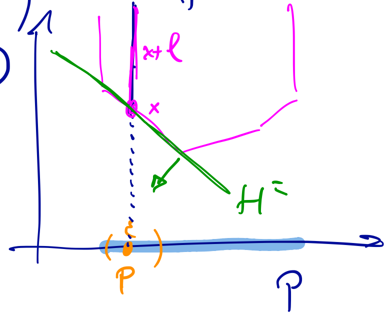
(2) $|\tilde{\tau}| = P$ "$\subseteq$" follows from $\pi(P^{\uparrow h}) = P$

For "$\supseteq$": enough to prove that interior of P is in $\pi(P^{\uparrow h})$

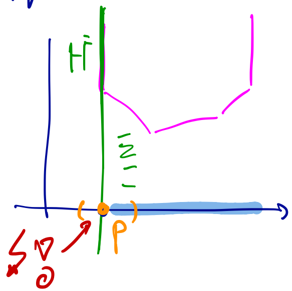
By contraposition: pick $p \in int(P)$, $\varepsilon > 0$ s.t. $B_\varepsilon(p) \subseteq int(P)$

Let x s.t. $x + \ell = P^{\uparrow h} \cap (p + \ell)$. x on boundary of $P^{\uparrow h}$,

belongs to a face supported on $H^=$



If $H^=$ is vertical:

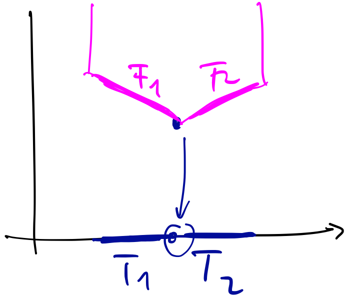


Thus x is in a bounded face $\Rightarrow$ $\overset{\text{"}p\text{"}}{\pi(x)} \in |\tilde{\tau}|$

(3) $\mathcal{T}$ is a polytopal complex.

Must show: for every $T_1, T_2 \in \mathcal{T}$,

$T_1 \cap T_2$ is face of both

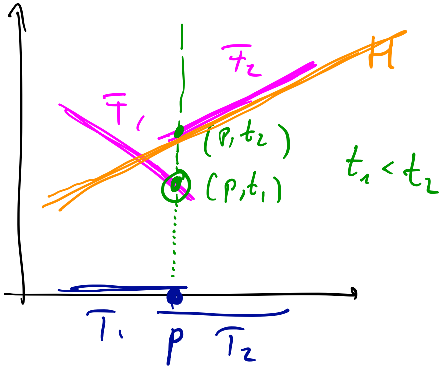


Enough to show: $\pi(\mathcal{F}_1 \cap \mathcal{F}_2) = T_1 \cap T_2$

- $\subseteq$ is trivial, since $T_i = \pi(\mathcal{F}_i)$
- $\supseteq$ Suppose there is

  $p \in T_1 \cap T_2 \setminus \pi(\mathcal{F}_1 \cap \mathcal{F}_2)$.

unfortunate.



Then as in the picture consider

$(p, t_1) \in \mathcal{F}_1 \cap (p + \ell)$

$(p, t_2) \in \mathcal{F}_2 \cap (p + \ell)$, wlog $t_1 < t_2$

and look at H supporting hyp. of $\mathcal{F}_2$

Then $(p, t_1) \in H^{\geq}$ ↯

Corollary: Every polyhedron P can be triangulated with no new vertices!

Pf: Take V = vertices of P, apply Theorem. $\blacksquare$