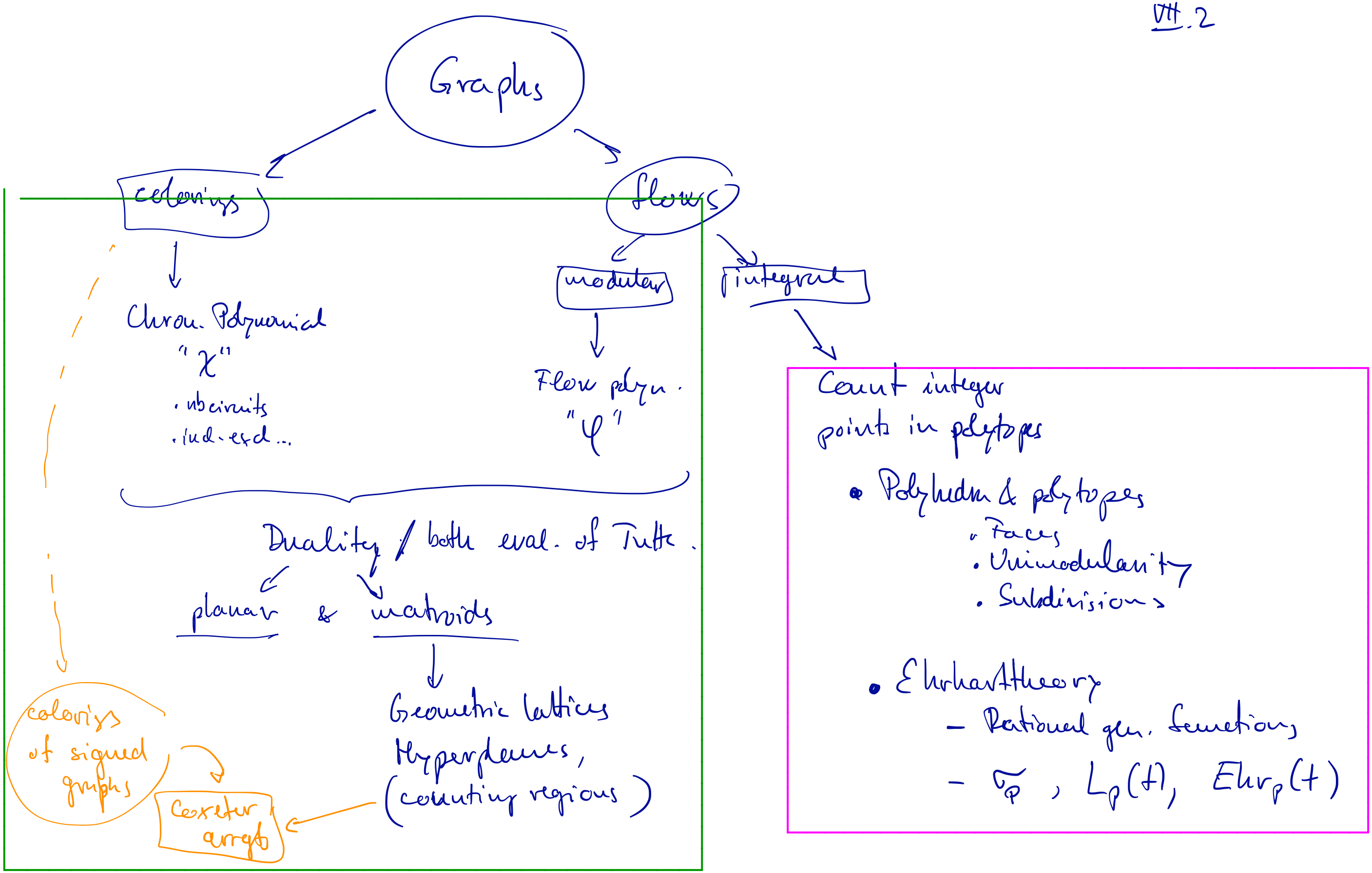
II.2

VI.1

VII.2



Graphs

colorings

Chrom. Polynomial

"χ"

- nbcircuits
- ind. set ...

flows

modular

Flow polyn.

"φ"

integral

Duality / both eval. of Tutte.

planar & matroids

Geometric lattices

Hyperplanes,

(counting regions)

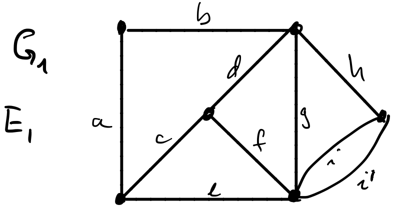
colorings of signed graphs

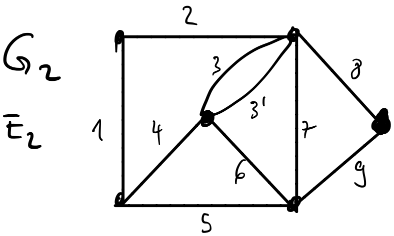
Coxeter arrgts

Count integer points in polytopes

- Polyhedra & polytopes
  - Faces
  - Unimodularity
  - Subdivisions
- Ehrhart theory
  - Rational gen. functions
  - σ_P, $L_P(t)$, $Ehr_P(t)$

II. 2





1) G_1, G_2 same Tutte polynomial. $T_{G_1}(x,y) = T_{G_2}(x,y)$

2) Non-isom. Matroids! $r_{G_i}: 2^{\{edges\}} \to \mathbb{N}$, $r_{G_i}(A) = \left\{\begin{matrix}\text{size of maximal} \\ \text{acyclic set in } A\end{matrix}\right\}$

CIRCUIT $C \subseteq \{edges\}$ with $r(C) = |C|-1$, $r(A) = |A|$ $\forall A \subsetneq C$

Assume isomorphism. i.e., Bijection $f: E_1 \longrightarrow E_2$ with $r_{G_1}(A) = r_{G_2}(f(A))$ $\forall A \subseteq E$

If such iso. exists: $\underbrace{f(C) \text{ is circuit in } G_2}$ iff $\underbrace{C \text{ circuit in } G_1}$.

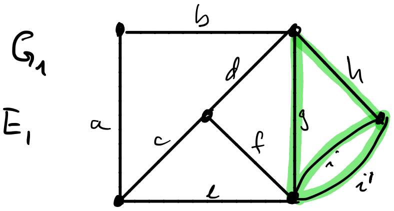
f bij

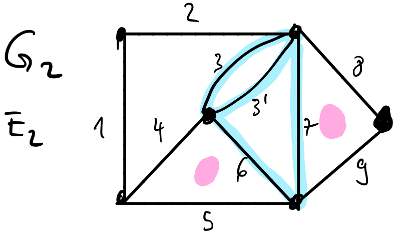
$r_{G_2}(f(C)) \overset{*}{=} |f(C)|-1 \iff r_{G_1}(C) = |C|-1$

f iso

$\forall X \subsetneq f(C)$ $r_{G_2}(X) = |X| \iff r_{G_2}(Y) = |Y|$ $\forall Y \subsetneq C$

II.2





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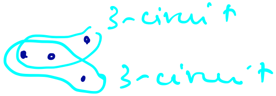
$f\left(g\,h\,\begin{matrix} i \\ i' \end{matrix}\right) =$



$\downarrow\downarrow$

$g, h \longmapsto 6, 7$ each of these is part of a 3-element circuit different from 🟦

only g is part of a 3-el. circuit outside of 🟩

$\boxed{\text{VI}.1}$ $a_0 = 2 \quad a_1 = 3 \quad \underbrace{a_n = 3a_{n-1} - 2a_{n-2}}_{\text{"(ii)"}}$ Find $\underbrace{\text{closed expr. for } a_n}_{\text{"(iii)"}}$

$\left(f(n) \sim a_n \right)$

α_i: coeff. of $f(\overset{\cancel{n}\,t+2-i}{\cancel{n}\,t+d-i})$

$\alpha_0 \qquad f(\cancel{n}\,t+2)$

$$\underset{\substack{\downarrow \\ \alpha_0 = 1}}{a_{\underset{t+2}{n}}} - \underset{\substack{\downarrow \\ \alpha_1 = -3}}{3a_{\underset{t+1}{n-1}}} + \underset{\substack{\downarrow \\ \alpha_2 = 2}}{2a_{\underset{t}{n-2}}} = 0 \quad \leftarrow d = 2$$

$$q(z) = 1 - 3z + 2z^2 = \underset{\substack{\downarrow \\ \underline{\gamma_1 = 2}}}{(1-2z)}^{\swarrow d_1 = 1}\underset{\substack{\downarrow \\ \underline{\gamma_2 = 1}}}{(1-z)}^{\swarrow d_2 = 1} \quad k=2$$

$$(\alpha_0 + \alpha_1 z + \alpha_2 z^2 \ldots)$$

Thm 6.2.1 Says:

$$a_n = \sum_{i=1}^{2} \underset{\nwarrow \text{degree} < d_i}{p_i(n)}\,\gamma_i^n$$

$$= x\gamma_1^n + y\gamma_2^n = x2^n + y$$

Use initial values:

$$\left.\begin{array}{l} a_0 = 2 \Rightarrow x + y = 2 \\ a_1 = 3 \Rightarrow 2x + y = 3 \end{array}\right\} \begin{array}{l} y = 1 \\ x = 3 - 2 = 1 \end{array}$$

$$\Rightarrow a_n = 2^n + 1$$

VI.1* $a_0 = 2 \quad a_1 = 3 \qquad a_n = 3a_{n-1} - 2a_{n-2} \quad \overset{n\geq 2}{\vee}$ Find closed expr. for a_n

Bare hands: $\Phi := \sum_{n\geq 0} a_n z^n$

$$\sum_{n\geq 2} a_n z^n = \underbrace{\sum_{n\geq 2} (3a_{n-1} - 2a_{n-2}) z^n} \qquad 3\left(\sum_{n\geq 2} a_{n-1} z^n\right) - 2\left(\sum_{n\geq 2} a_{n-2} z^n\right)$$

$$\Phi - 3z - 2 = 3\left(z \underbrace{\sum_{n\geq 1} a_n z^n}_{\Phi - a_0}\right) - 2\left(z^2 \underbrace{\sum_{n\geq 0} a_n z^n}_{\Phi}\right)$$

$$= 3z(\Phi - 2) - 2z^2\Phi = 3z\Phi - 6z - 2z^2\Phi$$

$$\Phi(1 - 3z + 2z^2) = 2 - 3z$$

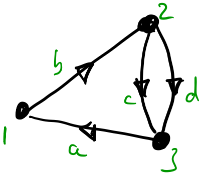
Partialbruch!

$$\Phi = \frac{2-3z}{(1-3z+2z^2)} = \frac{1}{(1-2z)} + \frac{1}{(1-z)}$$

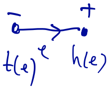
$$\Rightarrow \sum_{n\geq 0} a_n z^n = \sum_{n\geq 0} 2^n z^n + \sum_{n\geq 0} z^n \quad \Rightarrow \quad a_n = 2^n + 1$$

VII . 3

G



$I_G(k) = \#$ of positive k-flows $\left(\frac{(k-1)(k-2)}{2} \, ? \right)$



$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccc|c} a & b & c & d & \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 & 2 \\ -1 & 0 & 1 & 1 & 3 \end{array}$$

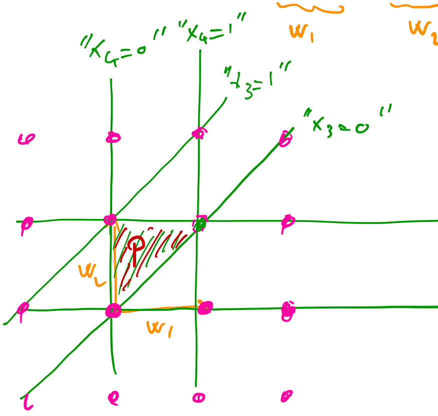
<u>Need</u>: $P = \{ \underbrace{Ax = 0}_{\ker(A)}, \underbrace{0 \le x \le 1}_{0 \le x_i \le 1,\ i=1,2,3,4} \}$

(really: we need $U \cap \mathbb{Z}^m$, and $P = \overline{U}$.)

$$\ker(A) = \left\langle \underbrace{\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{w_1}, \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}}_{w_2} \right\rangle =: W$$

Need to look at $W \cap \{x_1 \ge 0\}$, $W \cap \{x_1 \le 1\}$, ...

<u>Notice</u>: $W \cap \{x_4 = 0\}$ is a <u>line</u> in the "plane" W

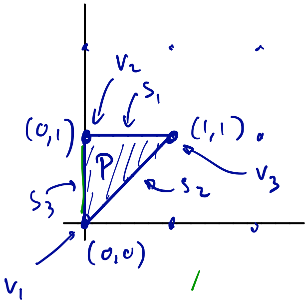


$W \cap \{\underline{x_1 = 0}\} = \langle w_1 \rangle = W \cap \{\underline{x_2 = 0}\}$ $\quad W \cap \{x_1 = 1\} = w_2 + \langle w_1 \rangle = W \cap \{x_2 = 1\}$

$W \cap \{x_3 = 0\} = \langle w_1 + w_2 \rangle$, $W \cap \{x_3 = 1\} = w_2 + \langle w_1 + w_2 \rangle$

$W \cap \{x_4 = 0\} = \langle w_2 \rangle$, $W \cap \{x_4 = 1\} = w_1 + \langle w_2 \rangle$

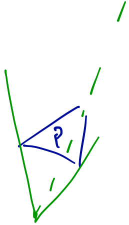
Note $\mathbb{Z}^4 \cap W = \left\{ \text{integer vectors of form } \underbrace{\lambda_1 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{\lambda_1 \in \mathbb{Z}} + \underbrace{\lambda_2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}}_{\lambda_2 \in \mathbb{Z}} \right\} = \mathbb{Z}w_1 + \mathbb{Z}w_2$



From Lecture:

$$\mathrm{Ehr}_P(t) = \sigma_{C(P)}(1,1,z) = \frac{\sigma_{\Pi}(1,1,z)}{(1-z)^3}$$

with $C(P) = \mathrm{cone}\left\{ \underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{m_1}, \underbrace{\begin{pmatrix}0\\1\\1\end{pmatrix}}_{m_2}, \underbrace{\begin{pmatrix}1\\1\\1\end{pmatrix}}_{m_3} \right\}$



$$\Pi = \left\{ \lambda_1 \begin{pmatrix}0\\0\\1\end{pmatrix} + \lambda_2 \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda_3 \begin{pmatrix}0\\1\\1\end{pmatrix}, \ 0 \le \lambda_1, \lambda_2, \lambda_3 < 1 \right\}$$

integer points: $\lambda_2 \in \mathbb{Z}, \lambda_3 \in \mathbb{Z}, \lambda_1 \in \mathbb{Z}$

$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$

$$\sigma_{\Pi}(z_1, z_2, z_3) = \sum_{\underbrace{\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} \in \Pi \cap \mathbb{Z}^3}_{\text{only } \begin{pmatrix}0\\0\\0\end{pmatrix}}} z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} = 1$$

$$\Rightarrow \mathrm{Ehr}_P(t) = \frac{1}{(1-z)^3} = \left(\sum_{n \ge 0} z^n\right)^3 = \sum_{n \ge 0} \underbrace{\binom{n+2}{2}}_{= L_P(n)} z^n$$

$$\Rightarrow L_P(t) = \frac{(t+2)(t+1)}{2}$$

We need: $\boxed{\mathrm{int}(t \cdot P) \cap \mathbb{Z}^2} \longrightarrow L_P(t) - \underbrace{L_{s_1}(t)}_{(t+1)} - \underbrace{L_{s_2}(t)}_{(t+1)} - \underbrace{L_{s_3}(t)}_{(t+1)} + \underbrace{L_{v_1}(t)}_{1} + \underbrace{L_{v_2}(t)}_{1} + \underbrace{L_{v_3}(t)}_{1}$

In summary $I_G(t) = L_p(t) - 3(t+1) + 3$

$$= \frac{(t+2)(t+1)}{2} - 3t = \frac{t^2+3t+2-6t}{2}$$

$$= \frac{t^2-3t+2}{2} = \frac{(t-2)(t-1)}{2}$$